Modulated Gauge Theories and Fracton Behavior in 2D

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arXiv:2306.17121 and 2310.09490
(Claudio Chamon & Yizhi You)

Introduction

• One of the main goals in condensed matter theory is a complete classification and characterization of distinct phases of matter;

• Among others, symmetries are powerful tools in such endeavor;

$$\text{Symmetry groups } G \leftrightarrow \text{Classical phases of matter and } G\text{-SPTs;}$$

• Recently, exotic and richer phases of matter were also casted in terms of generalized notions of symmetries

$$\text{Higher-form symmetries} \leftrightarrow \text{Topological order}$$
$$\text{Subsystem symmetries} \leftrightarrow \text{Fracton order}$$
$$\vdots$$

Spatially Modulated Symmetries

• Subsystem symmetries can be thought of as special cases of Spatially Modulated Symmetries

$$G = \int d^d x \, f(\vec{r})\rho(\vec{r}), \qquad [G, P] \neq 0 \tag{1}$$

for P the generator of translations.

• Modulated symmetries can naturally constrain dynamics of excitations;

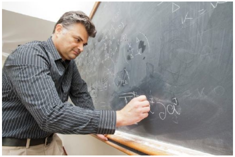
• Gauging dipolar symmetries $f(\vec{r}) = x, y, z$ was one of the first approaches towards EFTs for fractons in 3d (Pretko, M. (2017). PRB, 95(11), 115139.)

• In this talk, we are interested in exploring this idea for $\mathbb{Z}_N$ lattice systems and richer functions $f(\vec{r})$

Plan for the talk:

- Fractons and Motivations
- Topological order from Modulated Gauge Theories
- Exponentially symmetries as mechanism for fracton physics in 2d

Colaborators:



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Fracton Systems

Fracton systems usually share the common properties:

- (Gapped) Topological order $\rightarrow$ Long range entanglement;
- Restricted mobility of quasi-particles $\Rightarrow$ Sub-dimensional particles
  $\sim$ 0-dimensional particles $\Rightarrow$ Fractons;
  $\sim$ 1-dimensional particles $\Rightarrow$ Lineons;
  $\sim$ 2-dimensional particles $\Rightarrow$ Planons;
  $\Rightarrow$ "Fractional mobility!"
- Generalized symmetries (subsystem and modulated symmetries)
- UV/IR scales mixing

Fractons across dimensions

- In 3D, fractons systems usually present

$$\log GSD \sim L \tag{2}$$

- Theorem on homogeneous topological order in d dimensions (Jeongwan Haah. SciPost Phys. 10, 011 (2021))

$$\log GSD \leq c\mu L^{d-2} \tag{3}$$

- Conjecture for the non-existence of fracton phases in $d = 2$ (David Aasen et al. Physical Review Research 2.4 (2020): 043165)

- How "fractonic" a 2d system can be?
  $\sim \log GSD$ is $\mathcal{O}(1)$;

Spatially Modulated Symmetries

• Let us consider the lattice version of the modulated generators mentioned before

$$G[f] = \sum_{r} f_r \, q_r, \tag{4}$$

where q_r is some charge density and f_r is some lattice function at lattice position r.

• As examples, consider the familiar cases

$$\begin{aligned} f_r &= 1, && \text{Charge} \\ f_r &= x, y && \text{Dipole} \\ f_r &= x^2 - y^2, xy && \text{Quadrupole} \\ &\vdots && \\ f_r &= \text{n-th poly} && 2^n\text{-pole} \end{aligned} \tag{5}$$

$U(1)$ Modulated Symmetries

- We define $U(1)$ symmetry operators

$$U_\alpha[f] = \exp\left(i\alpha G[f]\right), \qquad \alpha \in [0, 2\pi) \tag{6}$$

- Action on bosonic matter fields:

$$b_r \to U_\alpha[f]\, b_r\, U_\alpha^\dagger[f] = e^{i\alpha f_r} b_r, \tag{7}$$

where the action depends on the lattice position r.

- Consider bosonic Hamiltonians that are invariant under under $U_\alpha[f]$.

<u>Example:</u> If $f_r = 1$ is the constant function, then

$$H_{\text{matter}} = -t \sum_r b_r b_{r+e_1}^\dagger + \dots \tag{8}$$

Matter System

- In general, the range of interaction depends on the function(s) f_r we choose. Schematically:

$$H_{\text{matter}} = -t \sum_r \prod_{a_r} b_{a_r}^{\Delta_{a_r}} + \dots \tag{9}$$

where the Δ_{a_r} are coefficients that define a "lattice derivative" operator

$$\sum_{a_r} \Delta_{a_r} f_{a_r} \equiv \Delta f \tag{10}$$

- The coefficients Δ_{a_r} are determined by the condition $\Delta f = 0$.

Symmetry	f_r	$\tilde{\Delta}_a$	$\prod_{a_r} b_{a_r}^{\Delta_{a_r}}$
Charge	1	$f_{r+e_a} - f_r$	$b_{r+e_a} b_r^\dagger$
Dipole	x, y	$f_{r+e_a} - 2f_r + f_{r-e_a}$	$b_{r+e_a} b_r^{\dagger\,2} b_{r-e_a}$
$\vdots$			
Exponential	m^{x+y}, for $m \in \mathbb{N}$	$f_{r+e_a} - m f_r$	$b_{r+e_a} b_r^{\dagger\,m}$

The condition for $\Delta f = 0$ follow from the invariance of H_{matter} under $U_\alpha[f]$

• Modulated gauge theory by requiring invariance under

$$b_r \to e^{i\alpha_r} b_r, \tag{11}$$

where α_r now depends on the lattice position.

• Introduce gauge fields A_a and E_a that minimally couple with the matter fields

$$H_{\text{matter+gauge}} = -\sum_{a,r} e^{-iA_{a,r}} \prod_{a_r} b_{a_r}^{\Delta_{a_r}} + \dots \tag{12}$$

$\sim$ Gauge transformations: $b \to e^{i\alpha} b$ and $A_a \to A_a + \Delta_a \alpha$,

Modulated Gauge Theories

• From the canonical relation$[A_{a,r}, E_{b,r'}] = i\delta_{a,b}\delta_{r,r'}$, the gauge transformations define a Gauss Law

$$q_r = \Delta_a E_a. \tag{13}$$

• One can also define a gauge invariant magnetic flux, schematically expressed as

$$b_r = \check{\Delta}_a A_a \tag{14}$$

where $\check{\Delta}_a$ is defined such that $\check{\Delta}_a \Delta_a \alpha = 0$ for any lattice function α_r.

$\sim$ It enforces that b_r is gauge invariant.

• The specifics of both Δ_a and $\check{\Delta}_a$ depend on the function f_r and the lattice symmetries one wishes to preserve.

$\mathbb{Z}_N$ Modulated Gauge Theories

• Integrating out the matter fields, we have two possibilities for the gauge fields:

$\Rightarrow$ Confined phase (Alexander M. Polyakov. Nuclear Physics B 120.3 (1977): 429-458).

$\Rightarrow$ Deconfined (Higgs) phase (More interesting one!)

• Higgs mechanism gap out the gauge fields and $U(1)$ is reduced down to $\mathbb{Z}_N \Rightarrow$ $\mathbb{Z}_N$ Lattice gauge theory!.

Deconfined phase of gauge theory $\Leftrightarrow$ Topological order

$\sim$ System still remembers conservation of $G[f]$ mod N through its Gauss law and mobility of anyons are constrained

$\mathbb{Z}_N$ theory

- $\mathbb{Z}_N$ lattice gauge $\leftrightarrow$ Impose Gauss-Law energetically;

- Higgs mapping: $e^{i2\pi E/N} = X$ and $e^{iA} = Z$. These are clock and shift operators! $XZ = e^{2\pi i/N} ZX$ ("$\mathbb{Z}_N$ Pauli matrices")

- Effective Hamiltonian

$$H_{\text{gauge bosons}} = -J_e \sum_r \underbrace{\mathcal{Q}_r}_{\text{Gauss-Law}} - J_m \sum_r \underbrace{\mathcal{B}_r}_{\text{Magnetic flux}} + \dots \tag{15}$$

with A_r and $B_{\tilde{r}}$ given, respectively, by

$$\begin{aligned} \mathcal{Q}_r &= e^{\frac{2\pi i q_r}{N}} = \prod_a X_a^{\Delta_a} \\ \mathcal{B}_r &= e^{\frac{i b_r}{N}} = \prod_a Z_a^{\check{\Delta}_a}. \end{aligned} \tag{16}$$

Properties

- By construction, these Hamiltonians are exactly solvable

$$\mathcal{B}_r \mathcal{Q}_r = \prod_a \omega^{-\check{\Delta}_a \Delta_a} \mathcal{Q}_r \mathcal{B}_r = \mathcal{Q}_r \mathcal{B}_r, \tag{17}$$

where we used that $\check{\Delta}_a \Delta_a = 0$.

$\Rightarrow$ This procedure provide us exactly solvable lattice realizations of $G[f_1] \oplus \ldots \oplus G[f_n]$ gauge theories

- The conservation laws are constraints for the eigenvalues of charge and flux operators

$$\prod_r \mathcal{Q}_r^{f_r} = \mathbb{1} \quad \text{and} \quad \prod_r \mathcal{B}_r^{\tilde{f}_r} = \mathbb{1}, \tag{18}$$

where $\tilde{f}_r$ are emergent flux conservation laws $\tilde{G}[\tilde{f}] = \sum_r \tilde{f}_r \, b_r$.

Several exactly solvable lattice models can be understood in this framework:

$$\mathbb{Z}_N \text{ Toric Code} \leftrightarrow G[1]$$

(Kitaev, A. Yu. Annals of Physics 303.1 (2003): 2-30.)

$$\mathbb{Z}_N \text{ Rank 2 Toric Code} \leftrightarrow G[1] \oplus G[x] \oplus G[y]$$

(Oh, Y. T., Kim, J., Moon, E. G., & Han, J. H. (2022). Physical Review B, 105(4), 045128.)

$$\text{Dipolar-Quadrupolar code} \leftrightarrow G[1] \oplus G[x] \oplus G[y] \oplus G[xy]$$

(**G.D**, Fontana, W., Gomes, P., & Chamon, C. (2023). SciPost Physics, 14(1), 002.)

$$\text{Moessner Code} \leftrightarrow G[1] \oplus G[x] \oplus G[y] \oplus G[x^2 + y^2]$$

(Benton, O., & Moessner, R. (2021). Physical Review Letters, 127(10), 107202.)

Subsystem Symmetries

• Subsystem symmetries have support only along sub-manifolds: spatially modulated!

• Consider a subsystem symmetry with support on a sub-lattice $\Omega \subset \Lambda$

$$\text{(Subsystem)} \quad \sum_{\Omega} q_r = \sum_{\Lambda} \delta_r(\Omega) q_r \quad \text{(Modulated)} \tag{19}$$

• Usual 3D fracton systems $\sim$ modulated gauge theories associated with

$$\oplus_i^N G[\delta(\Omega_i)], \tag{20}$$

where N is of the order $\mathcal{O}(L)$, with L the linear system size.

• How much of the particles' mobility can be constrained by requiring only $\mathcal{O}(1)$ symmetry generators?

Polynomial Symmetries

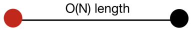
- First attempt: Gauge $G[f]$ for simple functions $f_r = \text{poly(x,y)}$

Let us consider, for example, dipolar symmetries $G[x]$ and $G[y]$ (Bulmash, D., & Barkeshli, M. (2018). Physical Review B, 97(23), 235112.)

$\sim$ Any process of particle creation has to conserve dipole moment, but only mod N

It means that N bosons can condense to the vacuum. Effectively, this introduces a length scale in the system of order N;

$\sim$ Particles can hop in steps of N sites.



$\Rightarrow$ Dipolar (and more generally, polynomial) symmetries apparently are not enough to fully constrain anyon mobility.

Exponential Symmetry

• Here, we consider generators of "**Exponential Symmetries**" (Sala, P., Lehmann, J., Rakovszky, T., & Pollmann, F. (2022). Physical Review Letters, 129(17), 170601.)

$$G[a^g] = \sum_r q_r \, a^{g_r}, \tag{21}$$

for a an integer mod N and g_r an arbitrary polynomial lattice function.

• Intuitively, exponential symmetries have a more powerful restriction on particle mobility

$$G[a^g] = \sum_r \sum_{n=0}^{\infty} \frac{1}{n!} \left(\ln a \, g_r\right)^n \, q_r = \sum_{n=0}^{\infty} \alpha_n \, G[g^n] \tag{22}$$

with $\alpha_n = \dfrac{ln(a)^n}{n!}$.

Exponential Symmetry Gauge Theory

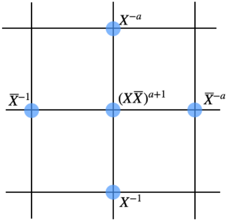
• Let us now consider the gauge theory associated with

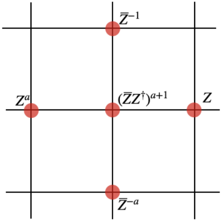
$$G[1] \oplus G[a^x] \oplus G[a^y] \oplus G[a^{x+y}] \tag{23}$$

• The procedure described previously provide us with the "toric-code-like" Hamiltonian

$$H = -\sum_r \mathcal{Q}_r - \sum_r \mathcal{B}_r, \tag{24}$$

with $\mathcal{Q}_r$ and $\mathcal{B}_r$ given, respectively, by





where there are two degrees of freedom per site.

Properties

- Exactly solvable Hamiltonian

- The ground state is gapped and topologically ordered. In periodic $L_x \times L_y$ lattice, is has degeneracy

$$\dim \mathcal{H}_0 = \big[N \gcd(N_a, a^{L_x} - 1) \gcd(N_a, a^{L_y} - 1) \gcd(N_a, a^{L_x} - 1, a^{L_y} - 1)\big]^2 , \tag{25}$$

where $N_a = \gcd(N, a)$.

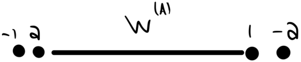
- **UV/IR mixing** $\Rightarrow$ Signal to fractonic physics!

- We can study the constrained dynamics of excitations through string operators

Restriction in Particle Mobility

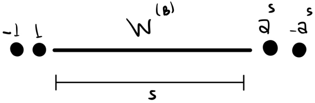
• There are two types of Wilson lines that are able to move dipolar bound states

$$W^{(A)} = \prod_{i=0}^{s} X_{r+ie_x} \tag{26}$$



∼ Ensured that $a \neq 0 \bmod N$, $W^{(A)}$ cannot be used to move single excitations.

$$W^{(B)} = \prod_{i=0}^{s} (X_{r+ie_x})^{a^i} \tag{27}$$



∼ $W^{(B)}$ cannot be used to move single excitations as well.

• Both are able to move only dipolar bound states but no single excitation.

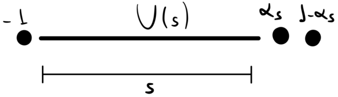
Moving single excitations

- Define the following function

$$\alpha_i \;=\; \sum_{j=0}^{i} a^j = 1 + a + \ldots + a^i \mod N \tag{28}$$

- Then, the following line operator might be able to move isolated excitations

$$U(s) = \prod_{i=0}^{s} (X_{r+ie_x})^{\alpha_i} \tag{29}$$



- If there exists an integer ℓ such that $\alpha_\ell = 1 \mod N$ then $U(\ell)$ can effectively move isolated excitations.

Remarks

• If there are no integer ℓ satisfying

$$\alpha_\ell = 1 \mod N, \tag{30}$$

there are no local strings able to move single particles $\Rightarrow$ signals towards type-I fracton physics

• The equation $\alpha_\ell = 1 \mod N$ has no solutions when

$$\gcd(a, N_a) \neq 1 \quad \Rightarrow \quad \text{No local strings condition} \tag{31}$$

• Since the GSD is finite, we expect the existence of non-local strings that are able to move single anyons

$\sim$ Non-local strings have support in $\mathcal{O}(L_x, L_y)$ sites and wind the system many times.

Tunneling time

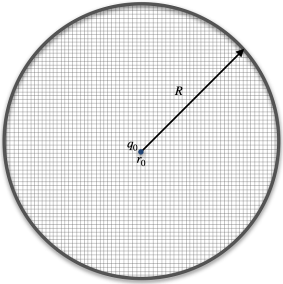
- Small perturbations to the Hamiltonian

$$H \to H - g \sum_r \left(X_r + X_r^\dagger \right), \qquad g \ll \text{Gap} \tag{32}$$

induce particles can hop;

$\sim$ Tunneling time (in perturbation theory) it takes for an isolated particle (in a region of radius R) to leave its position τ

$$\tau \sim \mathcal{O}(g^{-R})$$



- Hopping of isolated particles is protected against perturbations

Summary of Properties

Exponential Gauge Theory:

- ∼ Topological order ✓
- ∼ Fully immobile (single) particles ✓
- ∼ UV/IR mixing ✓
- ∼ Sub-extensive GSD growth ✗
- ∼ Membrane operators ✗

Topological order	**Symmetry generators**	**Immobile particles**	$\boldsymbol{\tau}$
2D Toric code	$G[1]$	NO	$\mathcal{O}(g^{-1})$
2D Rank-2 Toric code	$G[x] \oplus G[y]$	NO	$\mathcal{O}(g^{-N^2})$
2D Exponential gauge theories	$G[a^x] \oplus G[a^y]$	YES	$\mathcal{O}(g^{-R})$
3D X-Cube et.al.	$\oplus_i G[\delta(\Omega_i)]$	YES	$\mathcal{O}(R\, g^{-R})$

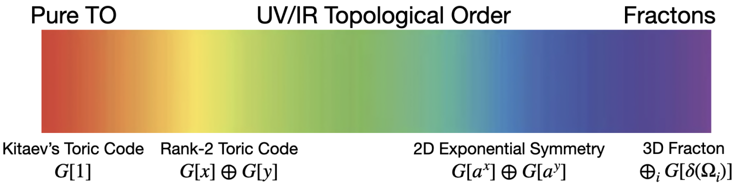
Final Considerations

• Modulated symmetries allow to map classification and characterization of quantum phases into function properties

• Although polynomial symmetries cannot restrict particles from hopping, exponential ones can

• Spatially modulated gauge theories, in general, are neither pure topological order nor fractons;

Instead, they belong to a continuum spectrum



Backup slides

Free Charged Bosons

• To warm up, let us consider a theory of charged bosons on a square lattice

$$H = -t \sum_r \left(b_r b^\dagger_{r+e_x} + b_r b^\dagger_{r+e_y} + \text{h.c.} \right) + \dots \tag{33}$$

∼ Here, $b^\dagger_r$ and b_r are the boson creation and annihilation operators at position r;

• The theory possesses a global $U(1)$ symmetry

$$b_r \to e^{if} b_r, \quad f \in [0, 2\pi) \tag{34}$$

associated to charge conservation;

∼ We now want to gauge the $U(1)$ symmetry by introducing gauge fields A_1 and A_2

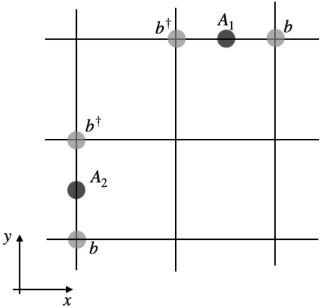
Gauging Global Charge

• The fields A_1 and A_2 sit at the edges of the lattice and the terms in the Hamiltonian are modified to

$$b_r e^{iA_{1,r+\frac{e_x}{2}}} b^\dagger_{r+e_x} + b_r e^{iA_{2,r+\frac{e_y}{2}}} b^\dagger_{r+e_y}. \tag{35}$$

Under gauge transformations

$$\begin{aligned} b_r &\to e^{if_r} b_r \\ A_{1,r+\frac{e_x}{2}} &\to A_1 + f_r - f_{r+e_x} \\ A_{2,r+\frac{e_y}{2}} &\to A_2 + f_r - f_{r+e_y} \end{aligned} \tag{36}$$



the Hamiltonian is invariant.

Properties

• Define the lattice derivatives along the x and y directions

$$\begin{aligned} D_1 f_r &\equiv f_r - f_{r+e_x} \\ D_2 f_r &\equiv f_r - f_{r+e_y} \end{aligned} \tag{37}$$

• We then define the gauge-invariant magnetic flux operator, which lives at the center of the plaquette on the dual lattice $\tilde{r}$

$$B_{\tilde{r}} = D_1 A_2 - D_2 A_1 \tag{38}$$

$\sim$ The gauge fields A_1 and A_2 have electric fields E_1 and E_2 as conjugate partners They obey

$$[A_{i,r}, E_{j,r'}] = i\delta_{ij}\delta_{r,r'}. \tag{39}$$

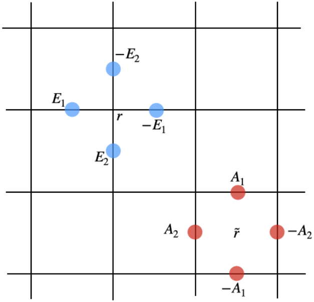
• The Gauss-Law is the generator of the gauge transformations. One can show that

$$q_r = D_1 E_1 + D_2 E_2 \tag{40}$$

• The vortices $B_{\tilde{r}}$ and the charges q_r from the bosonic theory are translated into defects in the gauge theory.

⇒ The Gauss Law and the gauge invariant flux define the $U(1)$ gauge theory

$$\begin{aligned} q_r &= D_1 E_1 + D_2 E_2 \\ B_{\tilde{r}} &= D_1 A_2 - D_2 A_1 \end{aligned} \tag{41}$$



• One can easily study the Maxwell-like dynamics of the system

$$H = \frac{1}{g_e^2} E_i E^i + \frac{1}{g_b^2} \cos(B) + \ldots \tag{42}$$

• Polyakov in 1977 showed that any such theory, with an underlying lattice, is always in the confined phase in 2+1 D (Polyakov, Alexander M. Nuclear Physics B 120.3 (1977): 429-458).

• Because of this, we are interested in the Higgs phase, where $U(1)$ is broken down to $\mathbb{Z}_N$ group.

• In the end, in the deep IR regime, we will get a topologically ordered phase - the $\mathbb{Z}_N$ lattice gauge theory

Discretizing to $\mathbb{Z}_N$ theory

• We now discretize the theory by energetically favoring the fields to assume values the N^{th} root of the unit circle

$$H \to H - U \sum_{i,r} \cos\left(\Delta_i \phi - N A_i\right) \tag{43}$$

which implements $\mathbb{Z}_N$ charges in the low-energy states of the theory.

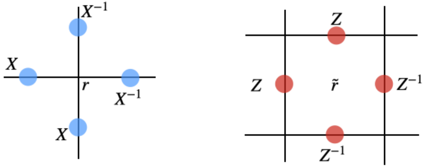
• As a result, the corresponding electromagnetic field can be expressed in terms of the $\mathbb{Z}_N$ Pauli matrices $e^{i\pi E} = X$ and $e^{iA} = Z$.

$$H = -J_e \sum_r A_r - J_m \sum_{\tilde{r}} B_{\tilde{r}} + \text{h.c.} + \dots \tag{44}$$

with

$$\begin{aligned} A_r &= X^{-1}_{r+\frac{e_x}{2}} X_{r-\frac{e_x}{2}} X^{-1}_{r+\frac{e_y}{2}} X_{r-\frac{e_y}{2}} \\ \text{and} \quad B_{\tilde{r}} &= Z^{-1}_{\tilde{r}+\frac{e_x}{2}} Z_{\tilde{r}-\frac{e_x}{2}} Z_{\tilde{r}+\frac{e_y}{2}} Z^{-1}_{\tilde{r}-\frac{e_x}{2}} \end{aligned} \tag{45}$$

- It consists of star and plaquette terms



$\sim$ This is precisely $\mathbb{Z}_N$ version of Kitaev's Toric code! (Kitaev, A. Yu. Annals of Physics 303.1 (2003): 2-30.)

- The conserved quantities (in a lattice with no boundary)

$$\sum_r q_r = 0, \quad \text{and} \quad \sum_{\tilde{r}} B_r = 0 \tag{46}$$

translate into the constraints

$$\prod_r A_r = \mathbb{1}, \quad \text{and} \quad \prod_{\tilde{r}} B_{\tilde{r}} = \mathbb{1}. \tag{47}$$

- Under periodic boundary conditions, these constraints lead to topological ground state degeneracy $GSD = N^2$

• To summarize, the correspondence between the matter theory and the gapped gauge theory are

Matter	$\leftrightarrow$	$\mathbb{Z}_N$ **Gauge Theory**
$U(1)$ global symmetry	$\leftrightarrow$	$\mathbb{Z}_N$ gauge invariance
Excitations	$\leftrightarrow$	Electric defects
Vortices	$\leftrightarrow$	Gauge flux
Mobility	$\leftrightarrow$	Wilson lines
Conservation laws	$\leftrightarrow$	Field constraints

• In the following, we propose a generalization of such gauge theories by gauging "**Exponential charge symmetries**".

$\sim$ We argue that this generalization may lead to the emergence of fracton-like phenomena in 2+1 dimensions.

• Here, we propose a special type of symmetry, that we dub "exponential polynomial"

$$\int q(r) a^{f(r)}\, dV \quad \leftrightarrow \quad \sum q_r a^{f_r}, \tag{48}$$

for an integer a mod N and f an arbitrary polynomial.

• These are not internal symmetries as they do not commute with generators of translation $[P, G] \neq 0$

Exponential Symmetry Gauge Theory

• Let us focus on one of the simplest case by allowing $f(x, y)$ to run over the choices of 1, x, y, and $x + y$;

$$\begin{aligned} \int q(r)\ dV &= G, \\ \int (a)^x q(r)\ dV &= G_x, \\ \int (a)^y q(r)\ dV &= G_y, \\ \int (a)^{x+y} q(r)\ dV &= G_{xy} \end{aligned} \tag{49}$$

• Based on this special conservation law, the possible charge dynamics on the lattice can be written as

$$b_r^{\dagger\, a} b_{r+e_x}^{a+1} b_{r+2e_x}^{\dagger} + b_r^{\dagger\, a} b_{r+e_y}^{a+1} b_{r+2e_y}^{\dagger} + \text{h.c.} + \dots \tag{50}$$

Properties

- After gauging,

$$A_1(r) \to A_1(r) + f(r+e_x) + af(r-e_x) - (a+1)f(r)$$
$$A_2(r) \to A_2(r) + f(r+e_y) + af(r-e_y) - (a+1)f(r) \tag{51}$$

- Define lattice "derivative" operators

$$\tilde{D}_1 = f(r+e_x) + af(r-e_x) - (a+1)f(r)$$
$$\tilde{D}_2 = f(r+e_y) + af(r-e_y) - (a+1)f(r)$$
$$D_1 = -f(r+e_x) - af(r-e_x) + (a+1)f(r)$$
$$D_2 = -f(r+e_y) - af(r-e_y) + (a+1)f(r) \tag{52}$$

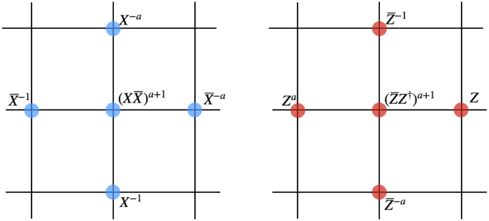
- We then have the generalized Gauss-Law and magnetic flux operators

$$D_1E_1 + D_2E_2 = q,$$
$$\tilde{D}_1A_2 - \tilde{D}_2A_1 = B \tag{53}$$

$\mathbb{Z}_N$ Theory

- Playing the same game as before and Higgsing the gauge theory down to Z_N, we have a Hamiltonian

$$H = -(X_r\bar{X}_r)^{a+1}(\bar{X}_{r+e_x})^{-a}(\bar{X}_{r-e_x})^{-1}(X_{r+e_y})^{-a}(X_{r-e_y})^{-1} - (Z_r^{-1}\bar{Z}_r)^{a+1}(\bar{Z}_{r+e_y})^{-1}(\bar{Z}_{r-e_y})^{-a}Z_{r+e_x}(Z_{r-e_x})^{a} + \text{h.c.} \tag{54}$$



where there are two degrees of freedom per site.

- For $a = 1$, one recovers a "BF" version of the dipolar Chern-Simons theory (Delfino, Guilherme, et al. SciPost Physics 14.1 (2023): 002).

Aspects of Number Theory

• First, we emphasize that the aforementioned protocol only works when $a_i \neq rad(N)$

Def.: Let $N = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$ be the prime decomposition of N, with p_i all prime numbers and $r_i \in \mathbb{N}$. We define $rad(N)$ as being the product of all the prime factors of N

$$rad(N) = p_1 p_2 \dots p_k. \tag{55}$$

• If $a = rad(N)$, one can infer that there exists a finite integer m such that $(a_i)^m = 0 \mod N$.

$\sim$ A simple solution is $m = \min(r_1, r_2, \dots, r_k)$.

• Consequently, the exponential symmetry only acts on a finite number of degrees of freedom for sites $|r| < m$ and cannot be further gauged

$$\sum_r q_r a^x = \sum_{|r|<m} q_r a^x \mod N. \tag{56}$$

Field Constrains

• Let N_a be the greatest divider of N that is coprime to a.

• Under periodic boundary conditions, the Higgsed conservation laws translate into the constrains

$$\prod_r A_r = \mathbb{1}, \quad \prod_{r=(x,y)} A_r^{\#_2(L)a^x} = \mathbb{1},$$
$$\prod_{r=(x,y)} A_r^{\#_2(L)\,a^y} = \mathbb{1}, \quad \prod_{r=(x,y)} A_r^{\#_3(L)\,a^{x+y}} = \mathbb{1}, \tag{57}$$

where $\#_i(L)$ are function that depend on the system size $L_x = L_y \equiv L$.

$\sim$ The flux operators B_r obey similar constraints.

• Such dependence comes from the requirement that the fields are compatible with the periodicity of the lattice - a manifestation of twisted boundary conditions.

Ground State Degeneracy

• The constrains on the charge and flux operators imply a ground state degeneracy rather exotic

$$\dim \mathcal{H}_0 = \big[N \gcd(N_a, a^{L_x} - 1) \gcd(N_a, a^{L_y} - 1) \gcd(N_a, a^{L_x} - 1, a^{L_y} - 1)\big]^2 . \quad (58)$$

• The IR information of the system depends sentitively on the lattice details. This is known as **UV/IR mixing**, and is very common in fracton systems

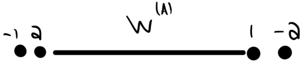
• The dependence of GSD on the system size is a direct consequence of $[P, G] \neq 0$, which is also a common property in fracton models

Question: Is this a fracton model?

Wilson lines moving charges

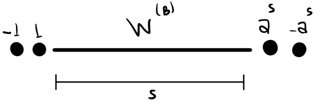
- There are basically two types of Wilson lines - A and B.

$$W^{(A)} = \prod_{i=0}^{s} X_{r+ie_x} \tag{59}$$



$\sim$ Ensured that $a \neq 0 \bmod N$, $W^{(A)}$ cannot be used to move single excitations.

$$W^{(B)} = \prod_{i=0}^{s} (X_{r+ie_x})^{a^i} \tag{60}$$



$\sim$ Ensured $a \neq rad(N)$, $W^{(B)}$ cannot be used to move single excitations as well.

- Both are able to move only dipolar bound states but no single excitation.

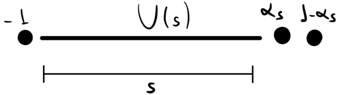
Moving single excitations

- Define the following function

$$\alpha_i \;=\; \sum_{j=0}^{i} a^j = 1 + a + \ldots + a^i \mod N \tag{61}$$

- Then, the following line operator might be able to move isolated excitations

$$\tilde{U}(s) = \prod_{i=0}^{s} (X_{r+ie_x})^{\alpha_i} \tag{62}$$



- Consider ℓ to be the smallest integer such that

$$\alpha_\ell = 1 \mod N. \tag{63}$$

Then when $s = \ell$, $U(\ell)$ can effectively move isolated excitations.

• We thus see that the commensurability among a and N introduces a length scale in the system.

• **Euler's totient Theorem:** When a and N are coprime, there always exists a finite integer $\varphi(N)$ such that $a^{\varphi(N)} - 1 = 0 \pmod N$

$\sim$ $\varphi(N)$ is known as Euler's totient function

$\sim$ In such case, the unit charge excitation can move up to $\varphi(N)$ sites along the x and y directions.

• When a and N are not coprime and $a \neq 0$ mod $rad(N)$, the equation $\alpha_\ell = 1 \pmod N$ does not necessarily have a solution.

$\sim$ No string operators able to move a single excitation

$\Rightarrow$ Type-I Fracton system in two-dimensions!

• For the case in which

$$\alpha_\ell \neq 1 \mod N. \tag{64}$$

for any integer ℓ, the model has no string operators able to move a single excitation

⇒ In such case, we have a fracton system in two-dimensions!

• It is interesting to note that $\alpha_\ell = 1 \bmod N$ is a variation of the ORDER-FINDING problem

$$a^p = 1 \mod N \tag{65}$$

∼ For large a and N, it is problem known to be computationally HARD.

∼ This brings up the idea that classifying a given phase of matter can be computationally hard! (Even for exactly solvable Hamiltonians)