Fractonic Behavior in Two Dimensions

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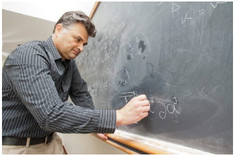
arXiv:2207.00409



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Some Recent History

The hype started around $\sim$ 2015, when people realized this was a new phase of matter (Vijay et al., 2015)

A New Kind of Topological Quantum Order:
A Dimensional Hierarchy of Quasiparticles Built from Stationary Excitations

Sagar Vijay, Jeongwan Haah, and Liang Fu
Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139

• "We term these fundamental excitations that behave as fractions of mobile particles 'fractons' ".(Vijay et al., 2015)

Before that, however, some models (containing fractons) were already known

- CBLT Code / Chamon Code (Chamon, 2005) + (Bravyi et al., 2011)

Quantum glassiness in clean strongly correlated systems: an example of topological overprotection

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- Haah Code (Haah, 2011)

Local stabilizer codes in three dimensions without string logical operators

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Institute for Quantum Information, California Institute of Technology, Pasadena, CA 91125, USA
(Dated: 28 February 2011)

Fracton Systems

Fracton systems usually share the common properties:

- (Gapped) Topological order $\rightarrow$ Long range entanglement;
- Restricted mobility of quasi-particles $\Rightarrow$ Sub-dimensional particles
  - $\sim$ 1-dimensional particles $\Rightarrow$ Lineons;
  - $\sim$ 2-dimensional particles $\Rightarrow$ Planons;

"Fractional mobility!"

- Exotic Symmetries / unusual conservation laws: higher-multipole momenta / subsystem symmetries;
- UV/IR scales mixing $\rightarrow$ $GSD \sim 2^L$

General Classification

There are two general classifications of fracton systems:

- **Type-I**: Isolated particles are immobile, but bound states can move.

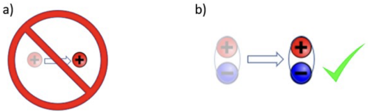


Figure: Figure borrowed from (Pretko et al., 2020).

- **Type-II**: All particles are immobile $\rightarrow$ There are no string operators.

Motivations and Possibilities

- Exotic physics:
  - ∼ Anyons in three dimensions;
  - ∼ Generalized symmetries;
  - ∼ UV/IR mixing: renormalization group ceases to apply;
- Finite temperature quantum memories (?):
  - ∼ Particles struggle to hop around the system;
  - ∼ Large ground state degeneracy 2^L;

What is possible?

- In 3D, fractons systems usually present

$$\log GSD \sim L \tag{1}$$

- Jeongwan Haah: Theorem on homogeneous topological order in d

dimensions (Haah, 2021)

$$\log GSD \leq c\mu L^{d-2} \tag{2}$$

- Other works corroborate the nonexistence of fractons in two dimensions (Aasen et al., 2020) + more;

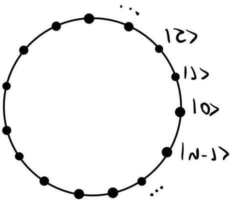
- Question: What does remain if we "project" 3D fracton physics in 2D?

Warm up: Wen Plaquette model (Wen, 2003)

- Z_N degrees of freedom at sites of a square $2D$ lattice

- Z_N clock and shift unitary operators $XX^\dagger = ZZ^\dagger = \mathbb{1}$ obey $XZ = e^{2\pi i/N} ZX$
$\rightarrow$ "Z_N Pauli algebra"

$$\sim Z^N = X^N = \mathbb{1}.$$

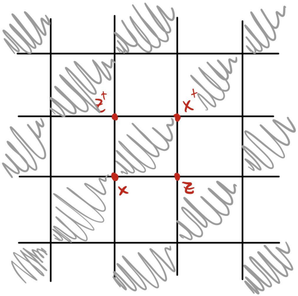


Define the plaquette operator as

$$\hat{F}_i \equiv X_i Z_{i+\hat{e}_1} X^\dagger_{i+\hat{e}_1+\hat{e}_2} Z^\dagger_{i+\hat{e}_2} \tag{3}$$

and the Hamiltonian

$$\hat{H} = -\sum_i \left(\hat{F}_i + \text{h.c.} \right) \tag{4}$$



Wen Plaquette Model

$\Rightarrow$ Exactly solvable;

- Eigenvalues of $F_i = e^{2\pi i p/N}$ for $p = 0, 1 \dots N-1$
  $\sim$ Ground states have eigenvalue $F_i = 1$ for all sites i.

- $L \times L$ periodic lattice, not all F_i eigenvalues are independent:

$$\prod_{i \in A} \hat{F}_i = \mathbb{1} \quad \text{and} \quad \prod_{i \in B} \hat{F}_i = \mathbb{1} \quad \text{for L even,}$$
$$\prod_{i} \hat{F}_i = \mathbb{1} \quad \text{for L odd} \tag{5}$$

$\sim$ Can be seen as charge conservation conditions

Hilbert space dimension is $N^{\#_{\text{sites}}}$ and there are $N^{\#_{\text{sites}} - \#_{\text{constrains}}}$ labels F_i

Ground state space is $N^{\#_{\text{constrains}}}$-fold degenerate:

$\rightarrow$ Locally indistinguishable

$\sim$ $GSD = N^2$ for L even

$\sim$ $GSD = N$ for L odd

$\langle \psi_0 | \hat{\mathcal{A}}_i \hat{\mathcal{B}}_j | \psi_0 \rangle = 0$

Excitations

Excitations above the ground state correspond to $F_i \neq 1$

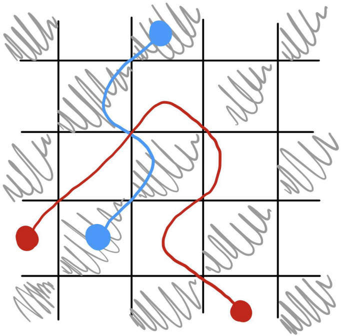
$\sim$ Gapped $\Delta = 4$

$\sim$ Created at the end points of strings

$\sim$ m and e excitations

$\sim$ mutual statistics: $\theta(m, e) = \frac{2\pi}{N}$

$\Rightarrow$ Anyons!



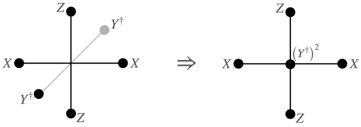
Effective field description: $U(1)_N \times U(1)_{-N}$ Chern-Simons gauge theory

$$S = \frac{N}{2\pi} \int_{T^2 \times \mathbb{R}} a \wedge db + J_e a + J_m b \quad \rightarrow \quad \text{TQFT!} \tag{6}$$

Quasi-Fractons[1] in 2D

General idea: "Project" a 3D fracton to 2D

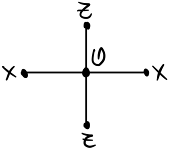
$\sim$ Dimensional reduction: 2D model obtained from CBLT model (Chamon, 2005; Bravyi et al., 2011)



We write the five-body interaction Hamiltonian as

$$H = -J \sum_{\vec{x}} \left(\hat{B}_{\vec{x}} + \text{h.c.} \right)$$

$\Rightarrow$ Topologically ordered!

$\hat{B}_{\vec{x}} =$  (7)

with

$$\mathcal{O}_{\vec{x}} \equiv \left(X^\dagger_{\vec{x}} \right)^2 \left(Z^\dagger_{\vec{x}} \right)^2 , \qquad (8)$$

[1]We first heard this from Salvatore Pace + Xiao-Gang Wen

Higher multipole momenta conservation!

When counting the *GSD*, we again identify the constrains among the plaquette operators. Important: $B_{\vec{x}}^N = \mathbb{1}$

$$
\begin{aligned}
\prod_{\vec{x}} \hat{B}_{\vec{x}} &= \mathbb{1}, \\
\prod_{\vec{x}=(\hat{x},\hat{y})} \left(\hat{B}_{\vec{x}}\right)^{\hat{x}\rho_x} &= \mathbb{1}, \\
\prod_{\vec{x}=(\hat{x},\hat{y})} \left(\hat{B}_{\vec{x}}\right)^{\hat{y}\rho_y} &= \mathbb{1}, \\
\prod_{\vec{x}=(\hat{x},\hat{y})} \left(\hat{B}_{\vec{x}}\right)^{\hat{x}\hat{y}\rho_{xy}} &= \mathbb{1}
\end{aligned}
$$

where

$$
\rho_a \equiv \frac{N}{\gcd(N, L_a)}
$$

and

$$
\rho_{xy} \equiv \frac{N}{\gcd(N, L_x, L_y)}.
$$

$\Rightarrow$ Dipole + off-diagonal quadrupole momenta conservation (mod Na and Na^2)

For every independent global constraint, the states have their degeneracy increased. In particular:

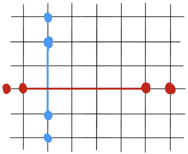
$$GSD = N \gcd(L_x, N) \gcd(L_y, N) \gcd(L_x, L_y, N) \quad (9)$$

∼ Locally indistinguishable states

∼ Robust against arbitrary perturbations

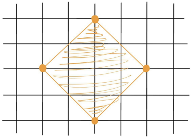
∼ Sensible UV/IR mixing

∼ Haah's upper bound is obeyed, as $\gcd(a, b) \leq a, b$



- String operators are rigid: dipoles are Lineons

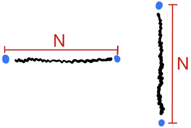
- To separate excitations apart, we need membrane operators: single excitations are (quasi-)fractons



One fundamental difference from Fractons:

• String of size N able to move isolated excitations - Non-uniform string

$\Rightarrow$ **Quasi-fractons**: Isolated particles



• The number of independent particles depends on the commensurability among L_x, L_y, and N.

The lengh-N operator redefines the step sizes that particles can hop:

$\sim$ They can now hop in steps of N

$\sim$ Particles that cannot be reached through string operators are "inequivalent" $\rightarrow$ Position-dependent anyons (Pace and Wen, 2022)

$\sim$ Excitations present position-dependent charge and mutual statistics, just as in (Oh et al., 2022b)

Emergence of time scale τ_{monopole}

The existence of the length-N operator implies in the emergence of a time scale. Perturbations induce particles to hop:

$$H \to H + g_x \sum_{\vec{x}} X_{\vec{x}} + \text{h.c.}, \quad g \ll 1 \tag{10}$$

While dipoles can hop in time

$$\tau_{\text{dipole}} \sim \left(\frac{J}{g}\right) \tag{11}$$

The hoping of a single particle from site $\vec{x}$ to $\vec{x} + N\hat{e}_2$ is obtained in $\sim N^2$th order in perturbation theory, implying a super-exponential characteristic time to hop

$$\tau_{\text{monopole}} \sim \left(\frac{J}{g}\right)^{N^2}. \tag{12}$$

Even though the model is not intrinsically fractonic, in practice one would never see a single excitation moving (even for mild values of N)

Effective Field Theorie (EFT) for $t \ll \tau_{\text{monopole}}$

The EFT description depends on the time scale the experiment is happening $t \ll \tau_{\text{monopole}}$ or $t \gg \tau_{\text{monopole}}$

For $t \ll \tau_{\text{monopole}}$, quasi-fractons effectively do not move. Physics is captured by Chern-Simons-like action

$$S_{CS} = -\frac{N}{2\pi}\int_{T^2\times\mathbb{R}} d^2l dt\left[A_1\partial_0 A_2 + A_0\left(D_1 A_2 - D_2 A_1\right)\right], \tag{13}$$

where $D_1 \equiv a\partial_x^2$ and $D_2 \equiv a\partial_y^2$.

$\sim$ Two-dimensional analogous of fractonic Chern-Simons theories in (You et al., 2020) and similar to (Oh et al., 2022a)

- There is a scale dimension a in the theory that cannot be removed

- Higher derivatives in EFT and gauge structures accounts for non-mobility of single excitations

Time scale $t \gg \tau_{\text{monopole}}$

- Effectively, all particles can hop normally
  - $\sim$ Higher-multipole momenta ($\sim$ mod Na, Na^2) ceases to be conserved
  - $\sim$ No length scale a appears

$$S = \frac{K_{IJ}}{4\pi} \int_{T^2 \times \mathbb{R}} a^I \wedge da^J, \tag{14}$$

with

$$K = \begin{pmatrix} N\sigma^x & 0_{2\times 2} \\ 0_{2\times 2} & N\sigma^x \end{pmatrix}. \tag{15}$$

- Correct GSD is obtained only lattice regularization
  - $\sim$ The UV/IR mixing is encoded in twisted boundary conditions

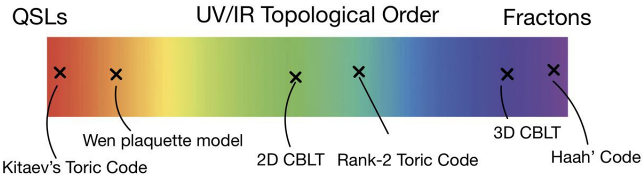
Final Remarks

• Multiple approaches to fracton order, not only spin systems (See for Higher-rank gauge theories (Pretko, 2018), (You et al., 2020), Generalized fusion theory (Pai and Hermele, 2019), etc.)

• UV/IR mixing and quasi-fractons emergent from "dimensional reduction" of fracton model

• Fine-tuned Hamiltonian ⇒ When we add more and more terms, what can remain and how sensible to the *UV* are such theories?

Sensibility of lattice geometry: Thanks to Salvatore Pace



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