Fusion Categories and SymTFT: Homework #1

Irreducible Representations of Finite Groups
January 23, 2026

Group and representation theory in its full generality typically warrants a full semester-long course. The purpose of this homework is to guide you through the basic steps involved in classifying the irreducible representations of a finite group. All concepts will be illustrated explicitly using the symmetric group S_3.

Problem 1: The symmetric group S_3

The symmetric group S_3 is the group of all permutations of three objects.

(a) List all elements of S_3. Write them in cycle notation (e.g. (123))
(b) What is the order $|S_3|$ of the group?

Problem 2: Conjugation and conjugacy classes

Definition. Two elements $g, h \in G$ are said to be *conjugate* if there exists an element $x \in G$ such that

$$h = xgx^{-1}.$$

The *conjugacy class* of an element g is the set

$$\mathrm{Cl}(g) = \{xgx^{-1} \mid x \in G\}.$$

You can show (optional) that conjugacy defines an equivalence relation on G. This means that the group is partitioned into several non-overlapping classes. These are important when finding the irreducible representations.

Now, consider the permutation group S_3:

(a) Determine the conjugacy class of the identity element.
(b) Consider the transposition (12).
   - Compute $x(12)x^{-1}$ for several choices of $x \in S_3$.
   - Determine the full conjugacy class of (12).
(c) Repeat the previous step for the 3-cycle (123).
(d) List all conjugacy classes of S_3. For each class, give:
   - a representative element,
   - the number of elements in the class.

Hint: In S_n, elements with the same cycle structure are conjugate.

Problem 4: Counting irreducible representations

A fundamental theorem in the representation theory of finite groups states:

Theorem. The number of inequivalent irreducible representations of a finite group equals the number of conjugacy classes. Using your results for S_3, how many irreducible representations does S_3 have?

Problem 6: One-dimensional representations of S_3

(a) Show that any one-dimensional representation of S_3 must map each group element to ± 1.
(b) Construct explicitly:
   - the trivial representation,
   - the sign representation.

(c) Verify that these two representations are inequivalent through unitary transformations.

Problem 7: The remaining irreducible representation

The dimensions d_i of the irreducible representations of a finite group satisfy

$$\sum_i d_i^2 = |G|.$$

(a) Use this formula to determine the dimension of the remaining irreducible representation of S_3.
(b) *(Optional)* All irreducible representation matrices are completely fixed—up to unitary equivalence—by the character table of the group. See Sections II.1–II.3 of Anthony Zee's Group Theory in a Nutshell for Physicists for a discussion of how character orthogonality relations determine these irreducible representations.

Problem 8: Irreducible representations of Abelian groups

(a) Use previous results to prove that all irreducible representations of a finite Abelian group are one-dimensional.
(b) Find all Irreps of the cyclic group $\mathbb{Z}_N$ in terms of roots of unity.