Fusion Categories and SymTFT: Homework #11

Quantum Double Model

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Problem 1: Vertex and plaquette operators in the quantum double model

Consider the operators A_s^g and B_p^h of the quantum double model $D(G)$, which satisfy the relations

$$(i) \quad A_s^{g_1} A_s^{g_2} = A_s^{g_1 g_2}, \tag{1}$$

$$(ii) \quad B_p^{h_1} B_p^{h_2} = \delta_{h_1,h_2} B_p^{h_1}, \tag{2}$$

$$(iii) \quad A_s^g B_p^h = B_p^{ghg^{-1}} A_s^g. \tag{3}$$

(a)

Define the operators

$$A_s := \frac{1}{|G|} \sum_{g \in G} A_s^g, \qquad B_p^e, \tag{4}$$

where e is the identity element of G.

Show that these operators are projectors, namely,

$$A_s^2 = A_s, \qquad (B_p^e)^2 = B_p^e. \tag{5}$$

(b)

Show that these operators commute:

$$[A_s, B_p^e] = 0. \tag{6}$$

Problem 2: Holonomies on a square lattice

Consider a two-dimensional square lattice in which each edge $e \in E$ is assigned a group element

$$\forall e \in E, \qquad g_e \in G. \tag{7}$$

Let γ be a loop on the lattice with a fixed counterclockwise orientation (image 1). Define the holonomy along γ by

$$\mathrm{Hol}(\gamma) = \prod_{e_i \in \gamma} g_{e_i}^{\epsilon_i}, \tag{8}$$

where

$$\epsilon_i = \begin{cases} +1, & \text{if the orientation of } e_i \text{ agrees with the orientation of } \gamma, \\ -1, & \text{otherwise.} \end{cases} \tag{9}$$

Perform the same calculation for another loop γ' and show that

$$\mathrm{Hol}(\gamma') = \mathrm{Hol}(\gamma_3)\,\mathrm{Hol}(\gamma_2)\,\mathrm{Hol}(\gamma_1), \tag{10}$$

where the loops γ_i correspond to elementary plaquette loops.

Furthermore, determine how $\mathrm{Hol}(\gamma)$ and $\mathrm{Hol}(\gamma')$ transform under the action of A_s^g.

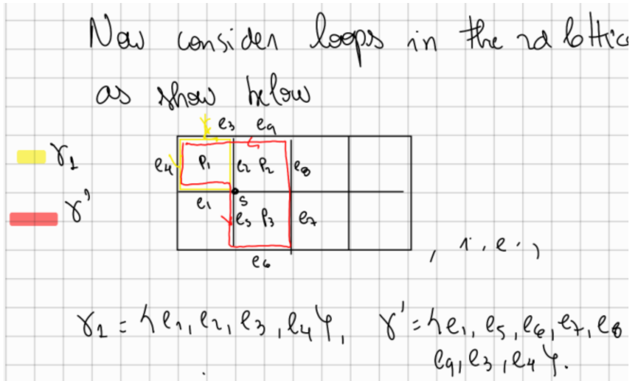

Figure 1: Example of loops on the lattice

Problem 3: Ground-state degeneracy on the torus

The ground-state degeneracy (GSD) of the quantum double model $D(G)$ for an Abelian group G is given by

$$\mathrm{GSD} = |\mathrm{Hom}(\pi_1(\Sigma), G)|, \tag{11}$$

where Σ is a two-dimensional surface.

For the torus $\Sigma = T^2$, we have

$$\pi_1(T^2) = \langle x, y \rangle, \tag{12}$$

where x and y are the two independent non-contractible loops.

Thus,

$$\mathrm{Hom}(\pi_1(T^2), G) = \{\rho : \pi_1(T^2) \to G\}, \tag{13}$$

that is, the set of group homomorphisms from $\pi_1(T^2)$ to G.

Show that

$$\mathrm{GSD} = |G|^2. \tag{14}$$

Problem 4: Anyon classification in the quantum double model

Given a finite group G, the anyons in the quantum double model $D(G)$ are classified by pairs

$$\alpha = (C, \rho), \tag{15}$$

where

- C is a conjugacy class of G,
- ρ is an irreducible representation of the centralizer Z_g, for some representative $g \in C$.

(a)

Calculate the number of anyons in D(G) considering G abelian, G = S_3 and G = $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ where i, j and k are imaginary numbers.