Fusion Categories and SymTFT: Homework #12

Quantum Double Model

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Exercise 1 — Toric-Code Operators from the Quantum Double of $\mathbb{Z}_2$

Consider the finite group $G = \mathbb{Z}_2 = \{e, g\}$. Define the operators on the local Hilbert space spanned by basis states $|x\rangle$, with $x \in G$, by

$$L_+^h |x\rangle = |hx\rangle, \qquad L_-^h |x\rangle = \left|xh^{-1}\right\rangle,$$

$$T_+^h |x\rangle = \delta_{h,x} |x\rangle, \qquad T_-^h |x\rangle = \delta_{h^{-1},x} |x\rangle.$$

For $G = \mathbb{Z}_2$, these simplify to

$$L_+^g = L_-^g =: L^g, \qquad T_+^e = T_-^e =: T^e, \qquad T_+^g = T_-^g =: T^g.$$

Now consider a square lattice with a fixed orientation. For each edge $e \in E$ we assign a local Hilbert space

$$\mathcal{H}_e = \mathrm{span}\{|e\rangle, |g\rangle\},$$

so each edge carries a qubit.

(a) Show that the operator L^g corresponds to the spin-flip operator σ^x.

(b) Show that

$$T^e + T^g = \mathbb{1},$$

and write σ^z in terms of T^e and T^g.

(c) Define the string operators

$$W(\bar{\gamma}) = \prod_{\bar{e} \in \bar{\gamma}} L_{\bar{e}}^g, \qquad J(\gamma) = \prod_{e \in \gamma} \left(T_e^e - T_e^g\right).$$

Show that

$$W(\bar{\gamma}) J(\gamma) = -J(\gamma) W(\bar{\gamma}),$$

assuming that the paths γ and $\bar{\gamma}$ intersect exactly once.

Exercise 2 — The S-Matrix of $D(G)$

Given two anyons $\alpha, \beta \in D(G)$, the modular S-matrix is given by

$$S_{\alpha,\beta} = \frac{1}{\mathcal{D}} \sum_{g \in C_a} \sum_{\substack{h \in C_b \\ gh = hg}} \chi_\rho(h)\, \chi_{\rho'}(g),$$

where

$$\alpha = (C_a, \rho), \qquad \beta = (C_b, \rho'), \qquad \mathcal{D} := \sqrt{\sum_\alpha d_\alpha^2}.$$

(a) Show that for two *electric* anyons (that is, anyons with trivial flux)

$$\alpha, \beta \in D(G),$$

the S-matrix satisfies

$$S_{\alpha,\beta} = \frac{d_\alpha d_\beta}{\mathcal{D}}.$$

(b) Consider $G = S_3$, and two *magnetic* anyons (that is, anyons with trivial charge)

$$D = (C_{(12)}, +), \qquad F = (C_{(123)}, +).$$

Show that

$$S_{DF} = 0.$$

Conclude that for any anyons of the form

$$\alpha = (C_{(12)}, \rho), \qquad \beta = (C_{(123)}, \rho'),$$

one has

$$S_{\alpha\beta} = 0.$$

Exercise 3 — The T-Matrix and Bosons/Fermions in $D(G)$

The modular T-matrix is diagonal and given by

$$T_{(C,\rho),(C',\rho')} = \delta_{CC'}\, \delta_{\rho\rho'}\, \theta_{(C,\rho)},$$

with topological spin

$$\theta_{(C,\rho)} = \frac{\chi_\rho(g_C)}{\chi_\rho(e)},$$

where g_C is a representative of the conjugacy class C.

If $\theta_{(C,\rho)} = 1$, we say the anyon is a boson, and if $\theta_{(C,\rho)} = -1$, we say the anyon is a fermion.

(a) Show that for any finite group G, every *electric* anyon is a boson.
(b) Show that for any finite group G, every *magnetic* anyon is a boson.
(c) Show that for $G = \mathbb{Z}_n$ with n odd, there are no fermions in $D(G)$.