Fusion Categories and SymTFT: Homework #2

Fusion Rings and Frobenius–Perron Dimensionss
March 8, 2026

Problem 1: The Fusion Ring of $\mathrm{Rep}(S_3)$

Consider the fusion ring given by the representation category

$$\mathrm{Rep}(S_3).$$

The set of simple objects is

$$\mathcal{L} = \{\mathbf{1}, \mathrm{sign}, \mathrm{std}\},$$

where $\mathbf{1}$ is the trivial representation, sign is the sign representation, and std is the two-dimensional standard representation.

The fusion rules are:

$$\begin{aligned} \mathrm{sign} \otimes \mathrm{sign} &= \mathbf{1}, \\ \mathrm{sign} \otimes \mathrm{std} &= \mathrm{std}, \\ \mathrm{std} \otimes \mathrm{std} &= \mathbf{1} \oplus \mathrm{sign} \oplus \mathrm{std}. \end{aligned}$$

(a) Define the dual object a^* of a simple object a by the condition that

$$\mathbf{1} \subset a \otimes a^*.$$

Determine the duals of $\mathbf{1}$, sign, and std.

(b) Using the fusion rules, decompose the following fusion-ring element in the basis $\mathcal{L}$:

$$\mathrm{sign} \otimes \mathrm{std} \otimes \mathrm{sign} \otimes \mathrm{std}.$$

(c) The *Frobenius–Perron (FP) dimension* d_a of a simple object a is defined as the largest positive eigenvalue of the fusion matrix

$$(N_a)_{bc} = N_{ab}^c,$$

where N_{ab}^c are the fusion coefficients. An alternative way to compute the FP dimension of simple objects is to consider the fusion rules $a \times b = \sum_{c \in L} N_c^{ab} c$, which imply a similar equation for the dimensions

$$d_a \, d_b = \sum_{c \in L} N_c^{ab} d_c. \tag{1}$$

This follows from the fact that the FP dimension defines a ring homomorphism.

(i) Compute the FP dimensions of all elements of $\mathcal{L}$.

(ii) Show that in $\mathrm{Rep}(S_3)$ the FP dimension of each simple object coincides with the ordinary dimension of the corresponding irreducible representation.

Problem 2: Constraints from Tensor Squares

Let τ be an irreducible representation of a finite group G.

(a) Is it possible for τ to satisfy

$$\tau \otimes \tau = \mathbf{1} \oplus \tau\,?$$

Give a clear argument for why this is possible or impossible.

(b) Explain why fusion rules of the form

$$\tau \otimes \tau = \mathbf{1} \oplus \tau$$

may appear in abstract fusion rings, but cannot arise from representation rings of finite groups.

Problem 3: The Haagerup Fusion Ring

[Adapted from Colleen Delaney's lecture notes]

Consider the **Haagerup** fusion ring, with label set

$$\mathcal{L} = \{\mathbf{1}, \omega, \omega^*, X, Y, Z\},$$

and fusion rules given by

$\times$	1	ω	ω^*	X	Y	Z
1	1	ω	ω^*	X	Y	Z
ω	ω	ω^*	1	Y	Z	X
ω^*	ω^*	1	ω	Z	X	Y
X	X	Z	Y	$1+X+Y+Z$	$\omega^*+X+Y+Z$	$\omega+X+Y+Z$
Y	Y	X	Z	$\omega+X+Y+Z$	$1+X+Y+Z$	$\omega^*+X+Y+Z$
Z	Z	Y	X	$\omega^*+X+Y+Z$	$\omega+X+Y+Z$	$1+X+Y+Z$

(a) Determine the dual object of each element of $\mathcal{L}$.

(b) Compute the Frobenius-Perron dimensions of the Haagerup fusion rules on the label set $L = \{1, \omega, \omega^*, X, Y, Z\}$.

(c) Can the Haagerup fusion rules correspond to the fusion rules for anyon?

Problem 4: Multiplicity-Free Fusion Rings

A fusion ring is said to be *multiplicity free* if all fusion coefficients satisfy

$$N_{ab}^{c} \in \{0, 1\} \qquad \text{for all } a, b, c \in \mathcal{L}.$$

(a) Show that in a multiplicity-free fusion ring, the fusion product of any two simple objects decomposes as a sum of *distinct* simple objects.

(b) Classify all multiplicity-free fusion rings of *rank 2*.
More precisely, let

$$\mathcal{L} = \{\mathbf{1}, X\}$$

and assume the fusion rules are multiplicity free. Determine all possible fusion rules consistent with associativity, the existence of a unit, and duals.