

Fusion Categories and SymTFT: Homework #3

Fusion Rings and Fusion Categories

March 8, 2026

Problem 1: Deligne product

[Adapted from Colleen Delaney's lecture notes]

- (a) Say we have two fusion rings, with label sets L_1, L_2 and fusion coefficients $N_{c_1}^{a_1 b_1}$ for $a_2, b_2, c_2 \in L_2$ and $N_{c_2}^{a_2 b_2}$ for $a_2, b_2, c_2 \in L_2$, respectively. **Check** that $L = L_1 \times L_2$ – which we will write as $L = \{a \boxtimes b\}_{a \in L_1, b \in L_2}$ – gives the free Z -module on L the structure of a fusion ring with respect to the multiplication on L given by $(a_1 \boxtimes a_2) \times (b_1 \boxtimes b_2) = \sum_{c_1 \in L_1, c_2 \in L_2} N_{c_1}^{a_1 b_1} N_{c_2}^{a_2 b_2} c_1 \boxtimes c_2$. (4 points)

We will see that this categorifies to a product of categories called the Deligne product, and that this operation can be interpreted as a stacking of physical theories.

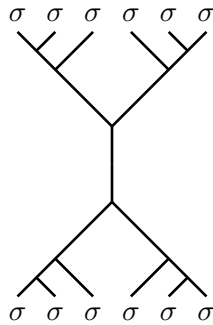
- (b) **Fill in the blanks** in the fusion table for the fusion ring $\text{Fib} \boxtimes \text{Ising}$.

		1 \boxtimes 1	1 \boxtimes σ	1 \boxtimes ψ	τ \boxtimes 1	τ \boxtimes σ	τ \boxtimes ψ
1 \boxtimes 1		1 \boxtimes 1	1 \boxtimes σ	1 \boxtimes ψ	τ \boxtimes 1	τ \boxtimes σ	τ \boxtimes ψ
1 \boxtimes σ		1 \boxtimes σ					
1 \boxtimes ψ		1 \boxtimes ψ					
τ \boxtimes 1		τ \boxtimes 1					
τ \boxtimes σ		τ \boxtimes σ					
τ \boxtimes ψ		τ \boxtimes ψ					

- (c) **Explain** why $d_{a \boxtimes b} = d_a d_b$ in any Deligne product of fusion rings, where d_a is the Frobenius-Perron dimension.¹ (2 points)

Problem 2: Admissible labelings

Enumerate the admissible labelings of this diagram using the Ising fusion rules. Assume all edges are oriented going down the page.



¹Recall from lecture that this notation d_a for $a \in L$ is typically reserved for the quantum dimension of an object (rather than the Frobenius-Perron dimension in either a unitary or spherical fusion category but that we're co-opting it anyway).

Problem 3: Morphisms and Trivalent graphs

Consider a fusion category \mathcal{C} with simple objects $\{a, b, \dots\}$, whose fusion rules are encoded by structure constants N_{ab}^c . Assume fusion multiplicities are finite.

Consider the vector space

$$\text{Hom}(a \otimes b, a \otimes b).$$

1. **Basis states.** Describe a basis for $\text{Hom}(a \otimes b, a \otimes b)$ in terms of admissibly labeled trivalent graphs. Your answer may be given as diagrams or as a description of all allowed intermediate labels.
2. **Dimension.** Show that the dimension of this vector space is

$$\dim \text{Hom}(a \otimes b, a \otimes b) = \sum_c N_{ab}^c N_c^{ab}.$$

3. **Special case.** Assume that all fusion multiplicities satisfy $N_{ab}^c \in \{0, 1\}$. Explain why $\dim \text{Hom}(a \otimes b, a \otimes b)$ is equal to the number of distinct fusion channels of $a \otimes b$.