Fusion Categories and SymTFT: Homework #4

Fusion Categories and $F-$symbol
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Problem 1: Triangle equations

Consider a multiplicity-free fusion category $\mathcal{C} = (L, \{N_c^{ab}\}, \{[F_c^{abc}]\})$. Draw the two natural choices for basis vectors in $\mathrm{Hom}(a \otimes b \otimes c, d)$ and consider the F-moves between such basis. Draw the pictures and argue why $[F_d^{a1c}]_{a,c} = 1$ and $[F_d^{ab1}]_{d,b} = 1$. These are called **triangle equations**.

Problem 2: Fusion Category $\mathcal{C} = \mathrm{Vec}_{\mathbb{Z}_N}^{\omega}$

Let $G = \mathbb{Z}_N = \langle g \,|\, g^N = e \rangle$ with multiplication

$$g^k\, g^q = g^{[q+k]_N} \tag{1}$$

where $k, q \in \{0, \ldots, N-1\}$ and $[q+k]_N$ denotes addition mod N. We define the fusion category $\mathrm{Vec}_{\mathbb{Z}_N}^{\omega}$ as follows:

- Simple objects are labeled by $a^k \in \mathbb{Z}_N$.
- Fusion rule is given by group multiplication

$$a^k \otimes a^q = a^{[q+k]_N}.$$

- The non-trivial $F-$symbols are given by:

$$\left[F_{d=abc}^{a,b,c}\right]_{m=ab,n=bc} = \omega(a,b,c).$$

where we take the explicit representatives

$$\boxed{\omega_n(a^k, a^q, a^p) = \exp\left(\frac{2\pi i n}{N^2}\, k(q + p - [q+p]_N)\right).}$$

with $n \in \{0, \ldots, N-1\}$ labels the representative.

(a) Show that $q + p - [q+p]_N$ is either 0 or N. Conclude that

$$\omega_n(a^k, a^q, a^p) = \begin{cases} 1 & q + p < N, \\ \exp\left(\frac{2\pi i n}{N} k\right) & q + p \geq N. \end{cases}$$

(b) Verify the normalization conditions:

$$\omega_n(a, b, 1) = \omega_n(a, 1, c) = \omega_n(1, b, c) = 1.$$

(c) Use the $F-$symbols given above and show that the Pentagon becomes

$$\omega_n(b,c,d)\omega_n(a,bc,d)\omega_n(a,b,c) = \omega_n(ab,c,d)\omega_n(a,b,cd).$$

Curious fact: The identity derived above is precisely the *3-cocycle condition*. It suggests that twisted fusion categories of the form $\mathrm{Vec}^{\omega}_{\mathbb{Z}_N}$ are naturally classified by group cohomology. More generally, consistent fusion categories $\mathrm{Vec}^{\omega}_{G}$ for a finite group G have their F-symbols determined by cohomology classes $\omega \in H^3(G, U(1))$ (you will prove this in the next homework). Such group cohomologies capture symmetry anoamalies and provides a classification framework for symmetry-protected topological (SPT) phases in $2+1$ dimensions.

(d) Use the explicit representative above and show that the Pentagon equation is in fact satisfied.

(e) Consider the explicit example of $N = 2$. Compute all values of $\omega_0(a,b,c)$ and $\omega_1(a,b,c)$. These are the two elements of

$$H^3(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$$

and both produce a consistent fusion category.