Fusion Categories and SymTFT: Homework #5

Fusion Categories and gauge transformations
March 8, 2026

[Adapted from Colleen Delaney's lecture notes]

Problem 1: 3-Cochains and 3-Cocycles

A 3-cochain on G with coefficients in $U(1)$ is a function $G \times G \times G \to U(1)$. The set of 3-cochains $C^3(G, U(1))$ forms a group under pointwise multiplication. A 3-cocycle on G valued in $U(1)$ is a function $\omega : G \times G \times G \to U(1)$ such that

$$\omega(gh, k, l)\omega(g, h, kl) = \omega(g, h, k)\omega(g, hk, l)\omega(h, k, l)$$

for all $g, h, k, l \in G$. **Check** that 3-cocycles on G form a subgroup of $C^3(G, U(1))$, call it $Z(G, U(1))$.

Problem 2: Pentagon Equation and 3-Cocycles

Let G be a finite group and consider the group ring $\mathbb{C}[G]$ as the fusion ring with basis $L = G$ and $N_k^{gh} = \delta_{k,gh}$ for all $g, h, k \in G$. **Show** that unitary[1] solutions to the pentagon equations for G-fusion rules over are 3-cocycles on G with $U(1)$ coefficients.

Problem 3: 3-coboundaries

A 3-cocycle ω is a 3-coboundary if there exists a function $\phi : G \times G \to U(1)$ such that

$$\omega(g, h, k) = \phi(g, hk)\phi(h, k)\phi(gh, k)^{-1}\phi(g, h)^{-1}$$

for all $g, h, k \in G$. Two 3-cocycles ω and $\tilde{\omega}$ differ by a coboundary if there exists a 3-coboundary ϕ such that

$$\omega(g, h, k) \cdot \tilde{\omega}(g, h, k)^{-1} = \phi(g, hk)\phi(h, k)\phi(gh, k)^{-1}\phi(g, h)^{-1}.$$

Check that differing by a 3-coboundary is an equivalence relation on G 3-cocycles with $U(1)$ coefficients.

Problem 4: Classification of $\mathrm{Vec}\, G$ fusion categories

Show that two unitary solutions to the pentagons for G-fusion rules are gauge equivalent if they differ by a 3-coboundary.

The group of equivalence classes of 3-cocycles modulo 3-coboundaries in called the third cohomology group of G with coefficients in $U(1)$ and is denoted by $H^3(G, U(1))$. These four exercises show that fusion categories with G-fusion rules are classified by $H^3(G, U(1))$.

[1] Here by unitary solutions all we mean are F-matrices which are unitary matrices.

Problem 5: Gauge Transformations

Suppose that some *multiplicity-free* F-symbols $[F_d^{abc}]_{m,n}$ are solutions to the pentagon equations. **Show** that an arbitrary gauge-transformation of these F-symbols also satisfy the pentagon equations.

Problem 6: Labeling anyons of Kitaev Quantum Double

Let $G = S_3$. Enumerate the pairs of the form (C, χ), where C is a conjugacy class in G and χ is an irrep of the centralizer subgroup $Z(c) = \{g \in G | gc = cg\}$ for a representative $c \in C$.

For example, take $C = \{(123), (132)\}$ and $c = (123)$. Then $Z(c) = \langle(123)\rangle \cong \mathbb{Z}_3, \ldots$ and so on.