Fusion Categories and SymTFT: Homework #6

Non-invertible symmetries in the Ising model and anyon chains
March 11, 2026

Problem 1: Non-locality of unitary U_{KW}

[Adapted from Sec. I of Xie Chen et al (2023)] Consider the $1+1$D transverse-field Ising model

$$H = -\sum_{i=1}^{N} Z_i Z_{i+1} - g \sum_{i=1}^{N} X_i, \tag{1}$$

where X_i, Z_i are Pauli operators acting on site i. The Hamiltonian has a global $\mathbb{Z}_2$ symmetry generated by

$$\eta = \prod_{i=1}^{N} X_i. \tag{2}$$

Define the operator

$$R(\mathcal{O}) = e^{-i\pi \mathcal{O}/4}. \tag{3}$$

For Pauli operators P, Q, the following identity holds:

$$R(Q) P R(Q)^\dagger = \begin{cases} P, & [P, Q] = 0, \\ iPQ, & \{P, Q\} = 0. \end{cases} \tag{4}$$

(a) Verify the above identity explicitly using the expansion

$$R(Q) = \frac{1}{\sqrt{2}}(1 - iQ), \tag{5}$$

together with $Q^2 = 1$.

(b) Define the two-site unitary

$$U_{i,i+1} = R(X_{i+1}) R(Z_i Z_{i+1}). \tag{6}$$

Using the conjugation rule from part (a), show that

$$U_{i,i+1}\, X_{i+1}\, U_{i,i+1}^\dagger = Z_i Z_{i+1}. \tag{7}$$

(c) Define the *sequential circuit*[1]

$$U_{KW} = R(Z_1 Z_N) \prod_{i=N}^{1} U_{i,i+1}, \tag{8}$$

where the product is ordered from right to left. Show that conjugation by U_{KW} maps

$$X_i \to Z_{i-1} Z_i, \qquad i = 2, \ldots, N. \tag{9}$$

Then compute explicitly the transformation of X_1 and show that it is mapped to the non-local string

$$X_1 \to X_1 X_2 \ldots X_{N-1} X_N \cdot Z_1 Z_N = \eta Z_1 Z_N \tag{10}$$

[1] There is a non-vanishing probability that this follows the inverse convention from the lecture notes.

(d) Similarly,compute the action of U_{KW} on Z operators and show that

$$\begin{aligned} Z_1 &\to Z_1 \\ Z_i &\to X_i X_{i+1} \dots X_N \cdot Z_1, \quad i = 2 \dots N \\ Z_i Z_{i+1} &\to X_i, \quad i = 2 \dots N \\ Z_1 Z_2 &\to X_2 \dots X_N = X_1 \eta \end{aligned} \tag{11}$$

Problem 2: Deformation of $1+1$d Ising chain

[Adapted from Shu-Heng Shao's lectures on 2024 Summer School @ IHES] Consider a 1D transverse-field Ising chain of length L with Pauli operators X_j, Z_j and periodic boundary conditions. In the presence of a λ deformation, the Hamiltonian is

$$H(g,\lambda) = -g\sum_{j=1}^{L} X_j - \sum_{j=1}^{L} Z_j Z_{j+1} + \frac{\lambda}{2}\sum_{j=1}^{L}\left(X_{j-1}Z_jZ_{j+1} + Z_{j-1}Z_jX_{j+1}\right). \tag{12}$$

The deformation preserves both the global $\mathbb{Z}_2$ symmetry and a non-invertible symmetry D (the latter is only a symmetry of the Hamiltonian at $g = g_c = 1$). This deformation was first studied by O'Brien and Fendley [ArXiv: 1712.06662] and is irrelevant for small λ, leaving the CFT intact. For sufficiently large λ ($\lambda > \lambda_c \approx 0.43$), however, the system becomes gapped (see figure below).

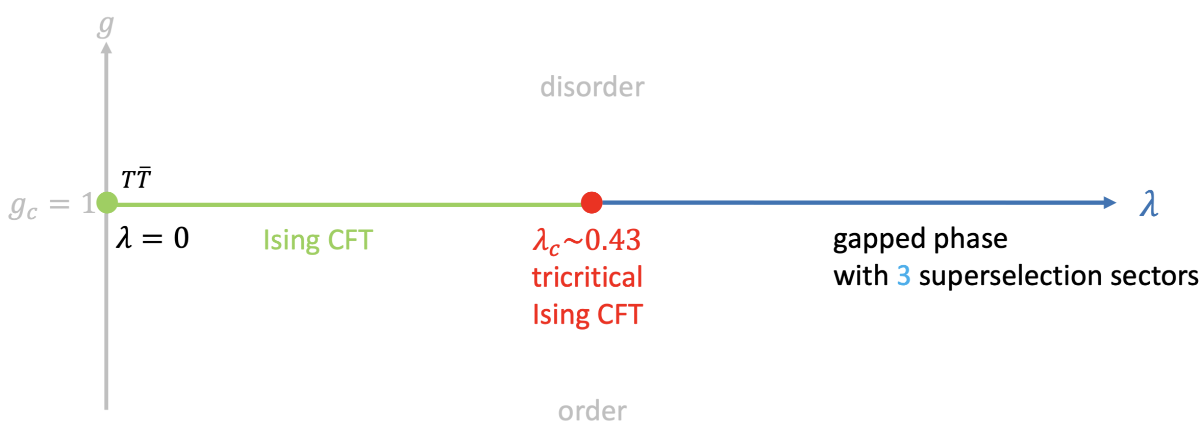

Figure 1: Phase diagram of Ising model with parameters g and λ. Figure taken from Shu-Heng Shao's talk @ KITP 2024.

In the following, you will show that for sufficiently large deformation parameter (specifically, $\lambda = 1$), the Hamiltonian $H(g = 1, \lambda = 1)$ has three gapped ground states given by

$$|++\cdots+\rangle, \quad |00\dots0\rangle, \quad |11\dots1\rangle. \tag{13}$$

(a) First, define the following local operators:

$$P_j^{(1)} = \frac{1}{2}(1 - Z_jZ_{j+1})(1 - X_{j-1}), \quad P_j^{(2)} = \frac{1}{2}(1 - Z_{j-1}Z_j)(1 - X_{j+1}). \tag{14}$$

Show that at $\lambda = 1$, the Hamiltonian can be written (up to an additive constant) as

$$H(g = 1, \lambda = 1) = \sum_{j=1}^{L}\left(P_j^{(1)} + P_j^{(2)}\right). \tag{15}$$

(b) Show that $P_j^{(1)}$ and $P_j^{(2)}$ are projectors, i.e., they are Hermitian and satisfy $P^2 = P$.

(c) Using the fact that projectors have eigenvalues 0 or 1, argue that the ground state energy of $H(\lambda = 1)$ is lower bounded by zero.

(d) Show explicitly that the following three product states

$$|++\cdots+\rangle, \quad |00\ldots0\rangle, \quad |11\ldots1\rangle \tag{16}$$

are annihilated by all $P_j^{(1)}$ and $P_j^{(2)}$, and therefore form exactly degenerate ground states of the Hamiltonian.

This corresponds to a gapped phase that spontaneously break the non-invertible symmetry D (as well as $\mathbb{Z}_2$).

Problem 3: Ising anyon chain

[Adapted from Colleen Delaney's lecture notes]

(a) Consider the anyon chain build on the Ising unitary *fusion* category with reference simple non-invertible object σ. Show that, under periodic boundary conditions the total Hilbert space $\mathcal{H}_{total}$ is only nontrivial for anyon chains with an even number of edges.

(b) Without computing formulas for how the symmetry generators D^1, D^σ, and D^ψ act on $\mathcal{H}_{total}$, show that $D^\sigma D^\sigma = D^1 + D^\psi$. *Hint:* how can you use the graphical calculus in a skeletal fusion category to resolve the fusion of two loops into the anyon chain into a fusion of a single loop?