Seminar UCL #1

Phases of matter

gapped vs gapless

Short range entangled (SRE) vs long range entangled (LRE)

Liquid vs non-liquid
↳ UV/IR (fractons)

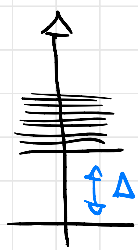
Category theory plays an important role!

Fusion Categories & SymTFT } Fri 14:00pm

- Groups, Representation theory, fusion rings, categories
- Unitary Fusion categories, Unitary Modular fusion categories
  - ↳ Generalized symmetries
  - ↳ Classify Topological order
- TQFTs, Turaev-Viro, Levin-Wen String Nets
- Drinfeld center, SymTFTs

⇒ Gapped. Hamiltonian H w/ eigenvalue E_n (thermodynamic limit)

⇒ Both SRE or LRE.



TQFTs

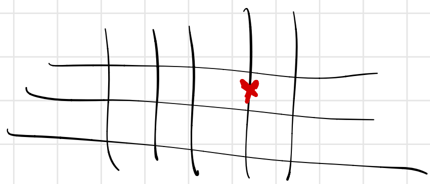
gapped SRE
- SSB phases
- SPTs phases

gapped LRE
- quantum spin liquids $\equiv$ Topological order
- FQHE
- ⋮

- Emergent anyons
- fractionalization
- top. entanglement entropy
- ground state deg. depending on topology

⋮

<u>Example:</u> Toric Code



$$H = -\sum A_s - \sum B_p$$

↪ Microscopic description

→ low energy states of TC's Hamilt. are topologically ordered.

Macroscopic + Universal

- No local order parameter

  ↳ Requires non-local measurements
- Anyons $1, \psi, e, m$ + fusion rules + braiding.

✗ • Gap value is NOT universal

↳ Fixed by global symmetries

Drinfeld center.

$\mathcal{Z}(\mathrm{Vec}\ \mathbb{Z}_2)$

$\vdots$

Phases given by Kitaev's quantum Doubles on $\mathcal{Z}(\mathrm{Vec}\ G)$

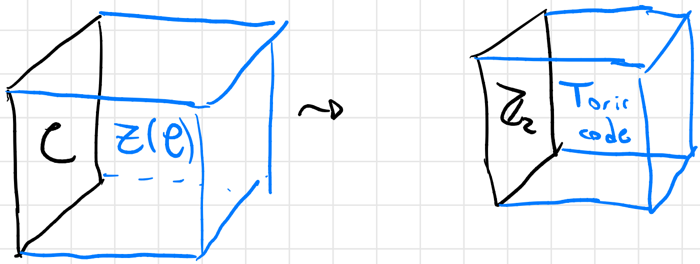
$\vdots$

$\mathcal{Z}(\mathcal{C})$

($\mathcal{C}$ fusion category)

$$\boxed{\text{fusion cat } \mathcal{C}} \xrightarrow[\text{center}]{\text{Drinfeld}} \boxed{\text{UMFC (top. order)}}$$

SymTFT (categorical holography)



Groups

Category theory naturally captures this.

Sets	homomorphism
Groups	isomorphism
Vector spaces	||
Anyons	$\vdots$
$\vdots$	

<u>Def.:</u> A group is a set G with a multiplication map

$$\cdot : G \times G \longrightarrow G$$
$$(g,h) \longmapsto g \cdot h .$$

i) $\exists$ multiplicative inverse $e \in G$, $e \cdot g = g \cdot e = g$ $\forall g \in G$

ii) $\exists$ inverses $\forall g \in G$ $\exists h \in G$ s.t. $g \cdot h = h \cdot g = e$ (Notation $h \equiv g^{-1}$)

iii) Associative $(g \cdot h) \cdot k = g \cdot (h \cdot k)$

Finite groups : $|G|$ = order of the group
↳ $|G|$ is finite.

<u>Examples:</u>

Cyclic group of order N :

$$g^2 = g \cdot g$$

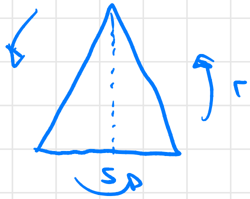
$$\mathbb{Z}_N = \langle g \mid g^N = e \rangle$$

- $\mathbb{Z}_N = \{ e, g, g^2, g^3, \dots, g^{N-1} \}$ → Abelian group.

• Dihedral group ⇝ symmetries of a n-gon

$$D_{2n} = \langle r, s \mid r^n = e,\ s^2 = e,\ rs = sr^{-1} \rangle$$

$n = 3$



(denoted D_n sometimes)

Non-Abelian group

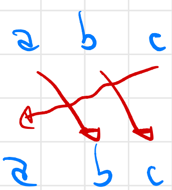
$$|D_{2n}| = 2n$$

• Permutation group S_n , $|S_n| = n!$

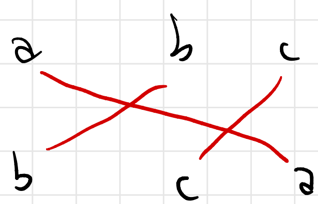
Eg: $n = 3$

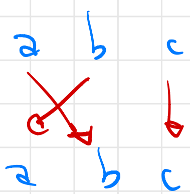
$$\begin{matrix} a & b & c \\ \downarrow & \downarrow & \downarrow \\ a & b & c \end{matrix}$$

⇝ identity permutation



~ cyclic permutation ≡ $(a\,b\,c)$





~ transposition

↳ Non-Abelian group.

Cayley's theorem: Every finite group G is a subgroup of S_n for some n.

Theorem: Every finite Abelian group G can be written as
$$G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s}$$

$\left\{ \begin{array}{l} \text{Subgroup,} \\ \text{isomorphisms} \\ \text{direct product} \\ \vdots \end{array} \right.$

Representation theory of finite groups

$\mathrm{Vec}\, G \simeq \mathrm{Rep}(G)$

Morita equivalent.

- When one gauges a finite symmetry G, quantum symmetries $\mathrm{Rep}(G)$ appear

↳ new symmetries in the gauged theory

- Kitaev Quantum Doubles: magnetic anyons labelled by group elements

electric anyons labelled by irreducible representations.

Def: Let G a finite group. A representation is a map

$$\rho : G \longrightarrow GL(n, \mathbb{C})$$

↳ group of $n \times n$ matrices over $\mathbb{C}$ with determinant 1.

with

$$\rho(g \cdot h) = \rho(g)\,\rho(h)$$

↳ "Represent" group elements by matrices

Ex: $G = \mathbb{Z}_2 = \{e, g\} \qquad g^2 = e$

$$\left.\begin{aligned} \mathrm{triv} : \mathbb{Z}_2 &\longrightarrow GL(1, \mathbb{C}) \\ g &\longmapsto 1 \end{aligned}\right\}$$ trivial representation triv, 1

$$\left.\begin{aligned} \mathrm{sgn} : \mathbb{Z}_2 &\longrightarrow GL(1, \mathbb{C}) \\ g &\longmapsto -1 \end{aligned}\right\}$$ sign representation sgn

Def: n in $GL(n, \mathbb{C})$ is the dimension of the representation

$$\rho_x : \mathbb{Z}_2 \to GL(2, \mathbb{C})$$
$$g \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2-dimensional representation.

$\vdots$

i) We can multiply representations (tensor product of matrices)

$$(\rho_1 \otimes \rho_2)(g) = \rho_1(g) \otimes \rho_2(g)$$

ii) We can add representations

$$(\rho_1 \oplus \rho_2)(g) = \rho_1(g) \oplus \rho_2(g)$$

the representations of a finite group G, together w/ addition $\oplus$ and multiplication $\otimes$ form a

Fusion ring

Def: A representation is called reducible if it can be written as a sum of smaller representations, up to a change of basis.

$$\rho_E : \mathbb{Z}_2 \longrightarrow GL(2, \mathbb{C})$$
$$g \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow \exists$ a unitary matrix U s.t.

$$U \rho_E(h) U^{\dagger} = \rho_X(h) \quad \forall h \in G$$

$$\rho_E(g) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) = \begin{pmatrix} \text{triv} & 0 \\ 0 & \text{sgn} \end{pmatrix} = \text{triv}(g) \oplus \text{sgn}(g)$$

$\Rightarrow \rho_E$ are reducible!

<u>Def</u>: Representations that are NOT reducible are called <u>irreducible</u>. Call these Irreps

<u>Remark</u>: All Irreps for Abelian groups are 1-dimensional
↳ Homework.

Ex: $G = S_3$ has three Irreps.

$$\mathbb{1}: S_3 \longrightarrow GL(1, \mathbb{C}) \qquad \text{trivial representation}$$
$$g \longmapsto 1.$$

$$\psi: S_3 \longrightarrow GL(1, \mathbb{C}) \qquad \text{sgn representation}$$
$$g \longmapsto \pm 1 \text{ (parity of transposition)}$$

$$\mathrm{std}: S_3 \longrightarrow GL(2, \mathbb{C})$$
$$(abc) \longmapsto \begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$$

Fusion table

$\otimes$	1	ψ	std
1	1	ψ	std
ψ	ψ	1	std
std	std	std	$1 \oplus \psi \oplus \mathrm{std}$.

$\rightsquigarrow$ fusion table of Kitaev Quantum Double $G = S_3$.

$$\mathrm{std} \otimes \mathrm{std} = 1 \oplus \psi \oplus \mathrm{std}.$$

Next: Fusion rings
Categories
Fusion categories.