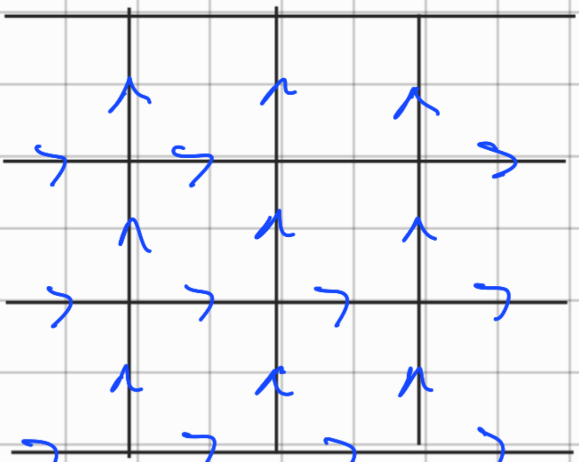
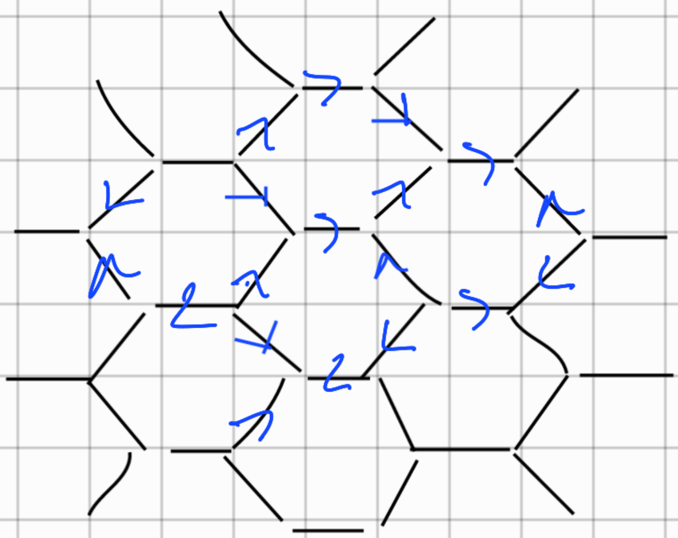
Quantum Double Model.

- G: finite Group
- (2+1)D lattice → planar.
  → Oriented Graph



→ — trivalent graph
  — square.

does not depend on the lattice geometry.



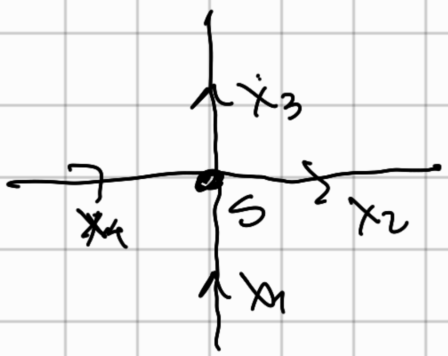
→ Orientation is a convention

- For each edge $e \in E$ hosts a Hilbert space $\mathcal{H}_i$, $\dim(\mathcal{H}_i) = |G|$

  Base: $|g\rangle$, $g \in G$.

- For each vertex

  $s \longrightarrow L_s$ a set of edges, i.e.,

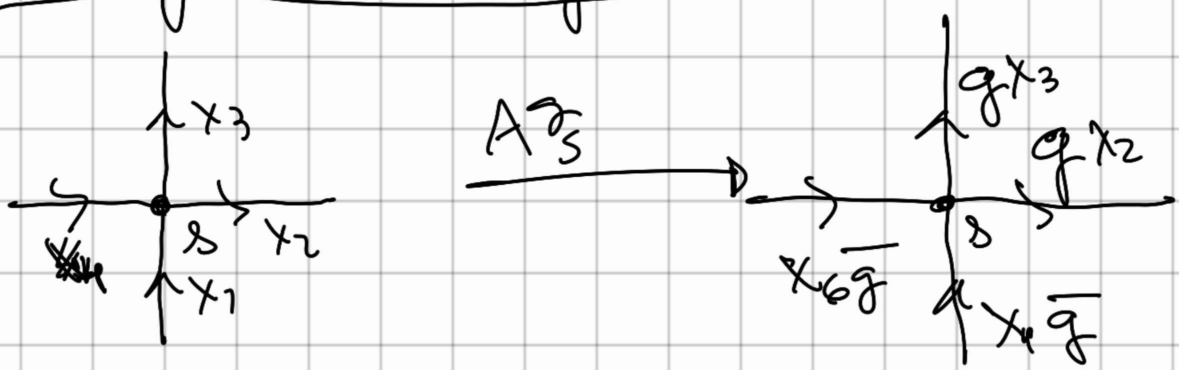


$$L_s = \{x_1, x_2, x_3, x_4\}$$

$$x_i \in G.$$

In this set we can act the star operator A^g_s, $g \in G$.

Diagramatically:



$$x_i \longrightarrow \begin{cases} g x_i & \text{if } x_i \text{ going out of } s \\ x_i \bar{g} & \text{if } x_i \text{ going into } s. \end{cases}$$

$$\bar{g} = g^{-1}.$$

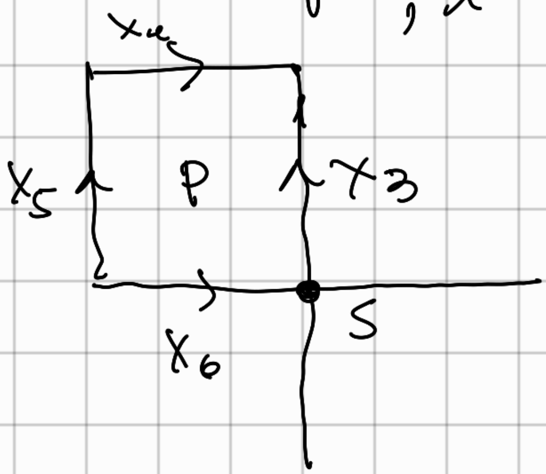
• Notation:

$$A^g_s |\underbrace{\ldots}_{\notin L_s}, x_1, x_2, x_3, x_6, \ldots\rangle =$$

$$= |\ldots, x_1\bar{g}, g x_2, g x_3, x_6 \bar{g}, \ldots\rangle$$

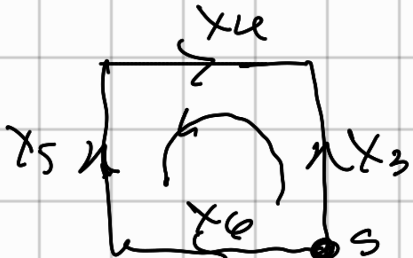
• $A^g_s \longrightarrow$ gauge transf.

• Given a face p, ∂p is a set of edges Γ_p, i.e.,

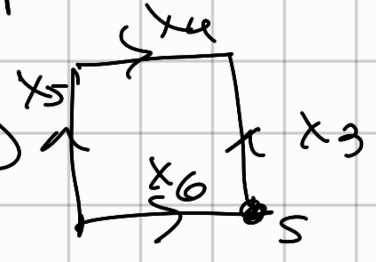


$$\Gamma_p = \{x_3, x_4, x_5, x_6\}$$

We can act the plaquette operator



$$\xrightarrow{B^g_p} \delta_{e, v(s,p)}$$



$U(s,p) = x_3 \bar{x}_4 \bar{x}_5 x_6$,

→ we start the concatenation in s in counterclockwise convention, and take the product $x_i (\bar{x}_i)$, x_i: aligned
$\bar{x}_i$: opposite.

Notation

$$B^h_{(s,p)} |\ldots, x_3, x_4, x_5, x_6, \ldots\rangle =$$

$$= \delta_{h, x_3 \bar{x}_4 \bar{x}_5 x_6} |\ldots, x_3, x_4, x_5, x_6, \ldots\rangle$$

$$= \delta_{h, U(s,p)} |\ldots\rangle$$

- B^h_p → project the holonomy (flux) on h.

The operators A^g_s, B^h_p satisfy:

(i) $A_s^{g_1} A_s^{g_2} = A_s^{g_1 g_2}$

Dem: $A_s^{g_1}\left(A_s^{g_2}|x\rangle\right) =$

$= A_s^{g_1}|\ldots, x_1\bar{g}_2, g_2 x_2, g_2 x_3, x_6 \bar{g}_2, \ldots\rangle$

$= |\ldots, x_1 \bar{g}_2 \bar{g}_1, g_1 g_2 x_2, g_1 g_2 x_3, x_6 \bar{g}_2 \bar{g}_1\rangle$

$= |\ldots, x_1 \overline{(g_1 g_2)}, g_1 g_2 x_2, g_1 g_2 x_3, x_6 \overline{(g_1 g_2)}\rangle$

$= A_s^{g_1 g_2}|x\rangle.$

$$\boxed{\therefore\ A_s^{g_1} A_s^{g_2} = A_s^{g_1 g_2}.}\quad \text{(i)}$$

(ii) $B_p^{h_1} B_p^{h_2} = \delta_{h_1, h_2} B_p^{h_1}.$

$\delta_{x,y}\,\delta_{z,y} = \begin{cases} 1, & x=y=z \\ 0 & \text{otherwise} \end{cases} = \delta_{xz}\delta_{xy}.$

Dem:

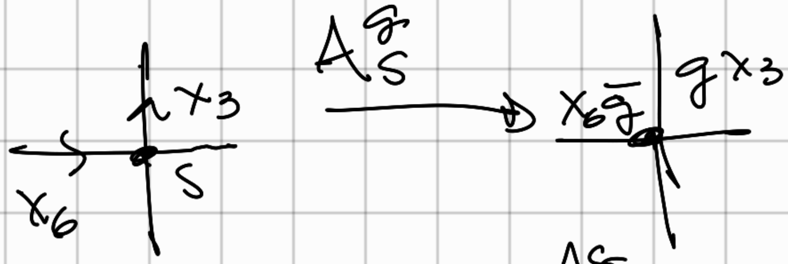
$B_p^{h_1} B_p^{h_2}|x\rangle = B_p^{h_1}\delta_{h_2, U(s,p)}|x\rangle$

$= \delta_{h_1, U(s,p)}\,\delta_{h_2, U(s,p)}|x\rangle$

$= \delta_{h_1, h_2}\,\delta_{h_1, U(s,p)}|x\rangle = \delta_{h_1, h_2} B_p^{h_1}|x\rangle$

(xiii) $A_s^g B_p^h = B_p^{ghg^{-1}} A_s^g$

<u>Dem</u>: (*) How $U(s,p)$ change under A_s^g.



$$U(s,p) \xrightarrow{A_s^g} g x_3 \cdots x_6 \bar g$$
$$= g\, U(s,p)\, \bar g.$$

<u>LHS</u>:

$$A_s^g B_p^h |x\rangle = A_s^g \delta_{h,U}|x\rangle = \delta_{h,U} A_s^g|x\rangle$$

<u>RHS</u>: $B_p^{gh\bar g} A_s^g|x\rangle =$

$$= B_p^{gh\bar g}|x'\rangle = \delta_{gh\bar g,\, U'(s,p)}|x'\rangle$$

$$= \underbrace{\delta_{gh\bar g,\, gU(s,p)\bar g}}_{\delta_{gh\bar g,\, gU\bar g} \,=\, \delta_{h,U}} A_s^g|x\rangle$$

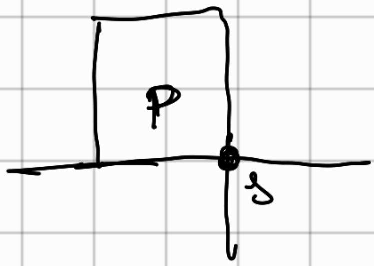
$$\Longrightarrow \delta_{h,U} A_s^g|x\rangle = \text{LHS}.$$

$$\boxed{\therefore\ A_s^g B_p^h = B_p^{gh\bar g} A_s^g.}$$

The operators A_s^g, B_p^h realize the $D(G)$ algebra: Drinfeld Double G (Hopf algebra)

- → <u>Def</u>: $D_{(g,h)} := A_s^g B_p^h$.

This is a algebra of local operator at site (s,p) on lattice.



- <u>Ground state sub space</u>

$V_{gs} = \mathrm{span}_{\mathbb{C}} \{ |\nu_{oc}\rangle \}$ satisfying

$A_s^g |\nu_{oc}\rangle = |\nu_{oc}\rangle$,

$B_p^h |\nu_{oc}\rangle = \delta_{h,e} |\nu_{oc}\rangle$, $\forall s,p$.
↳ $e \in G$ (neutral)

A Hamiltonian with such ground state

$$H_G = -\sum_s A_s - \sum_p B_p^e \qquad e : \text{neutral}$$

where

$$A_s := \frac{1}{|G|} \sum_{g \in G} A_s^g .$$

A_s and B_p^e are commuting projectors.

<u>Dem</u> $A_s^2 = A_s$.

$$A_s^2 = \frac{1}{|G|^2} \sum_{g \in G} A_s^g \sum_{h \in G} A_s^h =$$

$$= \frac{1}{|G|^2} \sum_{g,h} A_s^g A_s^h = \frac{1}{|G|^2} \sum_{g,h} A_s^{gh},$$

<u>But</u>: $\sum_{g \in G} \sum_{h \in G} A_s^{gh} = |G| \sum_{k \in G} A_s^k$.

Aside: $|G| = 3$

same structure

$$\sum_{g,h} A_s^{gh} = \sum_{gh} gh \,.\,,$$

take a $g = g_1$ (fixed)

$g_1 \sum_h h$ is a ~~permuta~~ permutation of $|G|$ elements, i.e.,

$|G| = n$

$$
\left.
\begin{array}{ll}
g_1(h_1, h_2, \dots, h_n) = (h_1', h_2', \dots, h_n') & \text{is a perm.} \\
g_2(h_1, h_2, \dots, h_n) = (h_1'', h_2'', \dots, h_n'') & \text{is } \cancel{\text{outher}} \text{ other perm} \\
\vdots & \\
g_n(h_1 \dots, h_n) = (h_1^n, h_2^n, \dots, h_n^n) &
\end{array}
\right\}
$$

~~total~~ total perm $|G| = n$

then $gh = k$ ~~appear~~ appear $|G|$ times

$$\Rightarrow \sum_{g,h} gh = |G| \sum_k k \,, \quad \text{such } k = gh.$$

$$\Rightarrow \sum_{g,h} A_s^{gh} = \sum_{k \in G} A_s^k \,.$$

Therefore:

$$A_s^2 = \frac{1}{|G|^2} \sum_{g \in G} \sum_{h \in G} A_s^{gh} =$$

$$= \frac{1}{|G|^2} |G| \sum_k A_s^k = A_s .$$

$$A_s^2 = A_s \quad \blacksquare$$

$$(B_p^h)^2 = B_p^h B_p^h = \delta_{h,h} B_p^h = B_p^h .$$

A_s, B_p^e are projectors!

$$[A_s, B_p^e] = 0 \quad \text{, since}$$

$$A_s B_p^e = \frac{1}{|G|} \sum_{g \in G} A_s^g B_p^e =$$

$$= \frac{1}{|G|} \sum_g B_p^{g e \bar{g}} A_s^g =$$

$$= \frac{1}{|G|} \sum_g B_p^e A_s^g = B_p^e A_s^g .$$

$$\Longrightarrow [A_s, B_p^e] = 0.$$

Since $[H_G, A_s] = [H_G, B_p^e] = 0$

The model is exactly solvable.

- Ground state

$A_s|vac\rangle = |vac\rangle$, $B_p^e|vac\rangle = |vac\rangle$

(sem carga) (sem fluxo)

eigenvalues of $A_s, B_p^e = \{0, 1\}$.

$$\Rightarrow \begin{cases} A_s|\psi\rangle = 0 & \text{(has an "electric" excitation on } s\text{)} \\ B_p^e|\psi\rangle = 0 & \text{(has an "magnetic" flux on } p\text{)} \end{cases}$$

Dimension of GS space (V_{gs})

$$\dim(V_{gs}) = |\,Hom(\pi_1(F_1), G)\,/\,G\,|$$

$Hom(X,Y)$: $\{\rho \mid \rho$ is a homomorphism of X in $Y\}$.

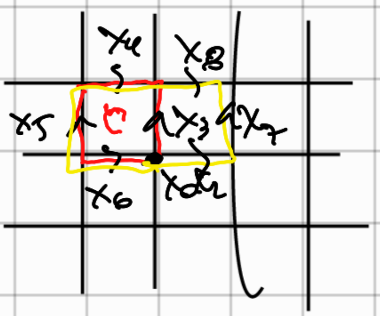
$\pi_1(F_1)$: fundamental group.

$\pi_1(F_1) = \{[\gamma],\ \gamma$ is a loop on $F_1\}$

$[\gamma] = \{\gamma'$ loop on F homotopic to $\gamma\}$.

• Intuição :

Dado uma variedade M, e uma curva γ



Σ (rede)

$\gamma = e_3 e_4 e_5 e_6$, e_i vertices

$\gamma' = e_2 e_7 e_8 e_4 e_5 e_6 \notin G$.

$Hol(\gamma) = x_3 \bar{x}_4 \bar{x}_5 x_6$

$Hol(\gamma') = x_2 x_7 \bar{x}_8 \bar{x}_4 \bar{x}_5 x_6$

Reescrever $Hol(\gamma') =$

$$= X_2 X_7 \bar{X}_8 \bar{X}_3 X_3 \bar{X}_4 \bar{X}_5 X_6$$

$= Hol(\gamma_1)\, Hol(\gamma)$ if we restrict to $V_{gs} \Rightarrow Hol(\gamma_1) = e$.

Since we impose vacum satisfy

$B_p^h |\text{Vac}\rangle = \delta_{h,e} |\text{Vac}\rangle \Rightarrow$

$Hol(\gamma)$ in a vacum state $= e$.

Therefore $Hol(\gamma) = e$

$$Hol(\gamma') = Hol(\gamma_1)\, Hol(\gamma') = e \cdot e = e.$$

Then

$$Hol(\Gamma) \longrightarrow G$$
$$\gamma \longmapsto Hol(\gamma) \in G$$

does not depend of γ if the one homotopic

$\Rightarrow$ We can construct the

$$\rho : \pi_1(\Sigma) \longrightarrow G$$
$$[\gamma] \longmapsto \mathrm{Hol}(\gamma).$$

This is a homomorphism.

But we ~~see that~~ $\mathrm{Hol}(\gamma)$ is conjugated under G action:

And the quotient space is leading with this,

then we see ~~that~~ we need to take the quotient space of

$\mathrm{Hom}(\pi_1(\Sigma_1), G) / G$ → remove the gauge equivalent holonomy.

For G abelian $\mathrm{Hol}(\gamma) \to g\,\mathrm{Hol}(\gamma)\,g^{-1} = \mathrm{Hol}(\gamma)$.

$\Rightarrow \dim(V_{\Sigma}) = |\mathrm{Hom}(\pi_1(\Sigma_1), G)|$

Exemple of Dim V_{gs} for Abelian G

- Consider $\Sigma = S^2$

$\pi_1(S^2) = \langle e \rangle$.

$\mathrm{Hom}(\pi_1(S^2), G) = 1$

There is only one homomorphism,

$\rho([0]) = e_G$

homomorphism must map $e_H \to e_G$. $\forall H, G$.

Therefore $\dim(V_{gs}) = |\mathrm{Hom}(\pi_1, G)|$
$= 1$.

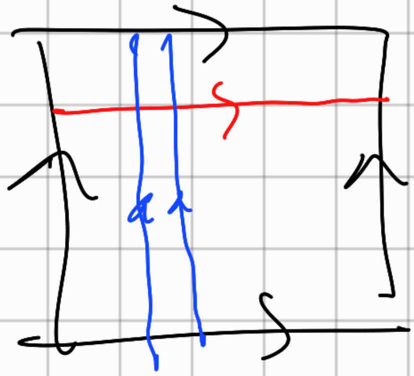
no degeneracy!

Ex Take $\Sigma = T^2 = S^1 \times S^1$

$\pi_1(\Sigma) \cong \mathbb{Z} \times \mathbb{Z} = a^m b^n, \; m, n \in \mathbb{Z}$.

i.e., $\mathbb{Z} + \mathbb{Z}$ has two generators a, b independent

a: loop on x direction
b: loop on y direction



Define

$\gamma = a b^2 = b^2 a.$

$$\rho : \pi_1(T^2) \longrightarrow G$$
$$a \longmapsto \rho(a) = g$$
$$b \longmapsto \rho(b) = h$$

Since ab generate all elements in $\pi_1(T^2)$ then we need just to know how is ρ on a, b.

$$\rho(ab) = \rho(a)\rho(b) = gh = \rho(ba) = hg$$
$$\Rightarrow gh = hg.$$

So if G is abelian $\forall g, h$ satisfy this

$$\Longrightarrow \mathrm{Hom}(\pi_1(T^2), G) \cong G \times G \Rightarrow$$
$$\Rightarrow |\mathrm{Hom}(\pi_1(T^2), G)| = |G \times G| = |G|^2$$

Therefore $\dim(V_{gs}) = |G|^2$,

<u>ex</u> $G = \mathbb{Z}_2$, $\dim(V_{gs}) = 2^2 = 4$

Anyon Content

Indexing irreducible representations (anyon content) of Drinfeld double
$G \quad D(G)$

<u>Hilbert space</u>

$$\mathcal{H} = \bigoplus_{\alpha} \mathcal{H}_\alpha \otimes \mathcal{H}_{\bar{\alpha}} ,$$

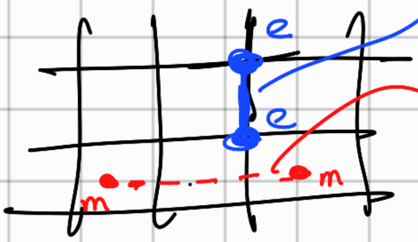
↳ antiparticle

α: anyon.

Decompose in superselection sectors.

The anyon is produce in same time with his antiparticle

→ <u>Toric code</u>:



line op. in original lattice.

line op in dual lattice

$m = \bar{m}$

$e = \bar{e}$

Each anyon type α of $D(G)$ is defined as following

(1) Conjugacy class of G

flux excitations

- Conjugacy class: Equivalence relation in a group G, given by:

$a \sim b \iff \exists g \in G : b = gag^{-1}$

Ex: Generic group G:

- $C(e) = \{ g \in G \mid g \sim e \}$

$g \sim e \iff \exists h \in G :$

$g = heh^{-1} = e$

$\Longrightarrow C(e) = \{e\} \ \forall G.$

Ex: G is abelian:

$C(g) = \{ h \in G \mid h = xgx^{-1} = \\ = g \}$

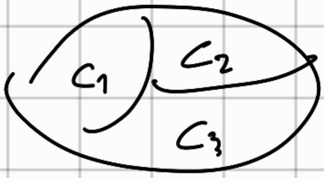
Therefore for G abelian

$$C(g) = \{g\} \quad \forall g \in G.$$

Aside:

Equivalence relation in a set $\longrightarrow$ Partition it in disjoint subsets

Ex: $G = S_3$



$C_i \cap C_j = \phi \quad i \neq j.$

(iii) An irrep of the centralizer representative member of the conjugacy class $C(g)$

$\longrightarrow$ "electric" charge

Aside: Centralizer of $g \in G$.

$$Z(g) = \{h \in G \mid hg = gh\}$$

$\longrightarrow$ is a subgroup of G

Ex: $\forall G, \; Z(e) = G$.

Ex: G abelian:

$$Z(g) = \{h \in G \mid gh = hg\} = G$$

Anyon $\longrightarrow (C(g), \rho)$

$C(g)$: conjugacy class

ρ : irrep$(Z(g') \mid g' \in C(g))$

- Consider G abelian

  - $C(g) = \{g\} \Rightarrow \# C(g) = |G|$

    therefore $\exists$ $|G|$ type of flux.

    take a representative of $C(g)$

    there is only one g.

    $\Rightarrow Z(g) = G$

    For abelian G $\#\,\text{irrep}(G) = |G|$

    Therefore $\exists$ $|G|$ type of charge

So the $D(G)$ for abelian G has

$|G|^2$ type of anyons

$= \#\,GSD(\Sigma = T^2)$

• For any G:

$C(e) \longrightarrow Z(e) = G.$

$\Rightarrow$ # $\neg$ flux (no flux) anyon =
= # irreps(G)

$\forall$ $C(g)$ we take irrep$\left(Z(g')\big|_{g' \in C(g)}\right) =$
= trivial

This correspond to a $\neg$ charge anyon

$\Rightarrow$ # conjugacy class =
= # $\neg$ charge anyon

But # conjugacy class =
= irreps(G) $\forall$ G (finite)

Therefore

$\neg$ flux anyons = # $\neg$ charge anyons

Ex: Consider $G = \mathbb{Z}_2 = \{1, m\}$

$|\mathbb{Z}_2|^2 = 4 \Rightarrow \exists$ 4 anyons.

$\#\text{ireps}(\mathbb{Z}_2) = 2$

$\Rightarrow$ 2 $\neg$ flux and 2 $\neg$ charge

but vacuum is $\neg$ charge and $\neg$ flux.

$\Rightarrow$ 1 vacuum, 1 $\neg$ charge, 1 $\neg$ flux

the other anyon has charge + flux.

$$\left.\begin{array}{l} \text{vacuum} \longrightarrow 1 \\ \neg\text{charge} \longrightarrow m \\ \neg\text{flux} \longrightarrow e \\ \text{charge} + \text{flux} \longrightarrow f \end{array}\right\}$$

Other way $D(\mathbb{Z}_2) = \{1, m\}$, $m^2 = 1$

$C(e) = \{e\} \longrightarrow Z(e) = \mathbb{Z}_2$

$C(m) = \{m\} \longrightarrow Z(m) = \mathbb{Z}_2$

Irreps $\mathbb{Z}_2$:

$$\chi_0(g) = e^{\frac{2\pi i \cdot 0 \cdot g}{2}} = 1 \quad \forall g$$

$$\chi_1(g) = e^{\pi i g}, \quad g \in \mathbb{Z}_2 = \{0,1\}$$

χ_0 : trivial : $+$

χ_1 : sgn : $-$

Anyons:

$(\{e\}, +) \to 1$

$(\{e\}, -) \to e$

$(\{m\}, +) \to m$

$(\{m\}, -) \to f.$

Fusion:

- Abelian G:
  - $C(g) = \{g\}, \ \forall g \in G$
  - $Z(g) = G \Rightarrow \mathrm{Irreps}(G), \ \forall g$:

anyon $\quad \alpha = (g, \rho), \ \rho \in \{\mathrm{Irreps}(G)\}$

$\beta = (g', \rho'), \ \rho' \in \{\mathrm{irreps}(G)\}$

$$\alpha \otimes \beta = (gg', \underbrace{\rho \otimes \rho'}_{\text{same group!}})$$

and ρ, ρ' are unidimentional.

$$\Rightarrow \alpha \otimes \beta = (\underbrace{g \circ g'}_{\text{group product}}, \underbrace{\rho\rho'}_{\text{usual multiplication in } \mathbb{C}})$$

Ex: $G = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$,

$a, b \in \mathbb{Z}_n$,

$a \circ b = a + b \mod n$.

irreps $(\mathbb{Z}_n)$,

$$\chi_k(g) = e^{\frac{2\pi i k g}{n}}, \quad k = 0, 1, \ldots, n-1$$

$$(\chi_k \chi_p)(g) = (\chi_k \otimes \chi_p)(g)$$

$$= e^{\frac{2\pi i (k+p) g}{n}} = \chi_{k+p}$$

Therefore we can write the $D(\mathbb{Z}_n)$ fusion rule by:

$$\alpha = (g, \chi_k) = (g, k) \quad , g, k = 0, 1, \dots, n-1$$

$$\beta = (g', \chi_{k'}) = (g, k')$$

And the fusion is given by:

$$\alpha \otimes \beta = (g, k) \otimes (g', k')$$
$$= (g + g' \bmod n, \; k + k' \bmod n)$$

Ex: $n = 2$ $\quad D(\mathbb{Z}_2) =$ Toric Code:

$(0,0) \rightarrow 1$
$(0,1) \rightarrow e$
$(1,0) \rightarrow m$
$(1,1) \rightarrow f$,

Aside:
of course for $G = \mathbb{Z}_n$
$\alpha \otimes \beta = \beta \otimes \alpha$
trivially!

$$e \otimes m = (0,1) \otimes (1,0) =$$
$$= (0+1, 1+0) = (1,1) = f.$$

$$f \otimes f = (1,1) \otimes (1,1) = (2 \bmod 2, 2 \bmod 2)$$
$$= (0,0) = 1$$

Then for $\alpha \in D(\mathbb{Z}_n)$,

$\alpha = (g, k)$, $\bar{\alpha} = (g', k')$ satisfy:

$g + g' \mod n = 0$

$k + k' \mod n = 0$.

Quantum dimension

the quantum dimension is given by:

for a $\alpha \in D(G)$

$\alpha = (C(g), \rho)$

$d_\alpha = |C(g)| \dim(\rho)$.

Ex: Consider G abelian

then $C(g) = \{g\} \Rightarrow |C(g)| = 1$

$Z(g) = G$, $\rho \in \mathrm{Irrep}(G)$

$\Rightarrow \dim \rho = 1$. ($G$ is abelian)

Therefore $d_\alpha = 1 \cdot 1 = 1$, $\forall \alpha \in D(G)$ α is abelian.

• Now lets take a look for a non abelian G.

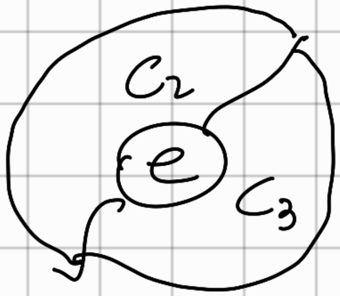
$$G = S_3 = \{e, (12), (13), (23), (123), (132)\}$$

Anyons Content

There are 3 conjugacy class:

$$C(e),\ C(12),\ C(123)$$

S_3 $\quad C_1,\ C_2,\ C_3$



$\Rightarrow \exists$ 3 type of flux.

Consider $C_1 = \{e\} \Rightarrow Z(e) = S_3$

S_3 has 3 ireps, $+$ (trivial) $\to$ dim 1
$-$ (signal) $\to$ dim 1
(2) (std) $\to$ dim 2

Then we have 3 anyons with trivial flux

$(\{e\}, +) \to A = 1$ (vacuum), dim 1

$(\{e\}, -) \to B$ ($\neg$ flux), dim 1

$(\{e\}, [2]) \to C$ ($\neg$ flux), dim 2

Now consider $C_2 = \{(12), (13), (23)\}$

take (12) as representative

$Z(12) = \mathbb{Z}_2$, $\mathbb{Z}_2$ has # ireps $= 2$

Anyons

$D = (C(12), +) \longrightarrow$ ($\neg$ charge), dim 3

$E = (C(12), -) \longrightarrow$ Dyon, dim 3.

Consider $C_3 = C(123) = \{(123), (132)\}$

$Z(123) \equiv \mathbb{Z}_3$, # ireps $\mathbb{Z}_3 = 3$

$(1, \omega, \omega^* = \omega^2)$

Anyons

$\omega(g) = e^{\frac{2\pi}{3} i g}$

$\hookrightarrow g = \{0, 1, 2\}$

$F = (C(123), +) \to$ ($\neg$ charge)

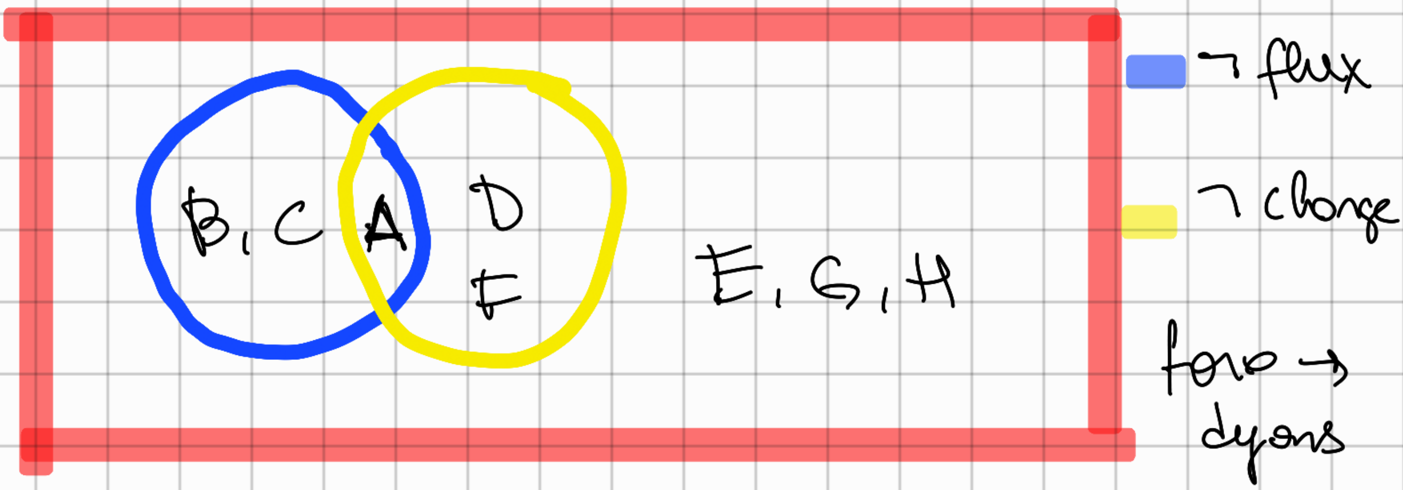
$G = (C(123), \omega) \longrightarrow$ (dyon)

$H = (C(123), \omega^*) \longrightarrow$ (dyon) $\Big\}$ dim 2

$\{A, B, C\} \to$ trivial flux

$\{D, E\} \to$ flux on class of odd perm.

$\{F, G, H\} \to$ flux on class of even perm.



Aside: Maybe (Warning!)

$$\mathcal{L}_e = \{A, B, C\}$$

$\to$ electric Lagrangian algebra

$$\mathcal{L}_m = \{A, D, F\}$$

$\to$ magnetic Lagrangian algebra.

C. class	Z	irrep	d	Label
C_1	S_3	$+$	1	A
		$-$	1	B
		$[2]$	2	C
$\lvert C_2\rvert = 3$ C_2	$\mathbb{Z}_2$	$+$	3	D
		$-$	3	E
$\lvert C_3\rvert = 2$ C_3	$\mathbb{Z}_3$	$+$	2	F
		ω	2	G
		ω^*	2	H

non Abelian anyons.

Fusion: Take the fusion of one element is of trivial flux as easy! But with elements where the two flux one in class $C(g) \neq C(e)$ we need caution!

Consider $\alpha = (C(e), \rho)$,

$\beta = (C(g), \rho')$

$$\alpha \otimes \beta = (C(e) \circ C(g), \rho \otimes \rho') =$$
$$= (C(g), \rho \otimes \rho')$$

But ρ is irrep of G and ρ' is irrep of $H \subseteq G$, then we need just restrict the irrep ρ on the subgroup $H \subseteq G$.

<u>Ex:</u>

$$C \otimes C = (C(e), [2]) \otimes (C(e), [2])$$

$$= (C(e), [2] \otimes [2]) =$$

$$= (C(e), [+] \oplus [-] \oplus [2]) =$$

$$(C(e), [+]) \oplus (C(e), [-]) \oplus (C(e), [2])$$

$$= A \oplus B \oplus C$$

Therefore $\boxed{C \otimes C = A \oplus B \oplus C}$.

Ex:

$$C \otimes D = \left(C(e), [2]\big|_{S_3}\right) \otimes \left(C(12), [+]\big|_{\mathbb{Z}_2}\right)$$

$$= \left(C(12), [2]\big|_{\mathbb{Z}_2} \otimes [+]\right)$$

We need write $[2]$ as sum of irreps of $\mathbb{Z}_2$. We use the character of irrep (which impose a constraint)

$\mathbb{Z}_2 = \{e, (12)\}$, $(12)^2 = e$.

$\chi_{[2]}(e) = 2$,

$\chi_{[2]}(12) = 0$,

then we have $\chi_{[2]}(e) = a\chi_{[+]}(e) + b\chi_{[-]}(e)$

$$\chi_{[2]}(12) = a\chi_{[+]}(12) + b\chi_{[-]}(12)$$

$$\Rightarrow \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} a \cdot 1 + b \cdot 1 \\ a \cdot 1 + b(-1) \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} 2 = a + b \\ 0 = a - b \end{array}\right\} \Rightarrow a = b = 1$$

Therefore $[2] = [+] \oplus [-]$

And we find

$$C \otimes D = \left((C(12)), [+] \right) \oplus \left(C(12), [-] \right)$$
$$= D \oplus E.$$

Ex: $B \otimes D = \left(C(e), [-] \right) \otimes \left(C(12), [+] \right)$

$$= \left(C(12), [-]\big|_{\mathbb{Z}_2} \otimes [+] \right)$$

$\mathbb{Z}_2 = \{e, (12)\}$, $\begin{pmatrix} \chi_{[-]_{S_3}}(e) \\ \chi_{[-]_{S_3}}(12) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a\chi_{[+]_{\mathbb{Z}_2}}(e) + b\chi_{[-]_{\mathbb{Z}_2}}(e) \\ a\chi_{[+]_{\mathbb{Z}_2}}(12) + b\chi_{[-]_{\mathbb{Z}_2}}(12) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \end{pmatrix} \Rightarrow \begin{matrix} a=0 \\ b=1 \end{matrix}$$

$$[+]_{S_3}\Big|_{\mathbb{Z}_2} = [-]_{\mathbb{Z}_2}.$$

Therefore $B \otimes D = \big(C(12), [-]\big) =$

$$\Rightarrow B \otimes D = E$$

- The lines A, B, C is only to construct!

But when no one of the anyon in fusion has trivial flux, we need caution, e.g.,

Ex: $D \otimes D = (C(12), +) \otimes (C(12), +)$

$= (C(12) \circ C(12), [+] \otimes [+])$

$$C(12) \circ C(12) = \bigcup_{g \in C(12)} (g \circ C(12)) =$$

$$= \{(12) \circ C(12)\} \cup \{(13) \circ C(12)\} \cup$$

$$\cup \{(23) \circ C(12)\} =$$

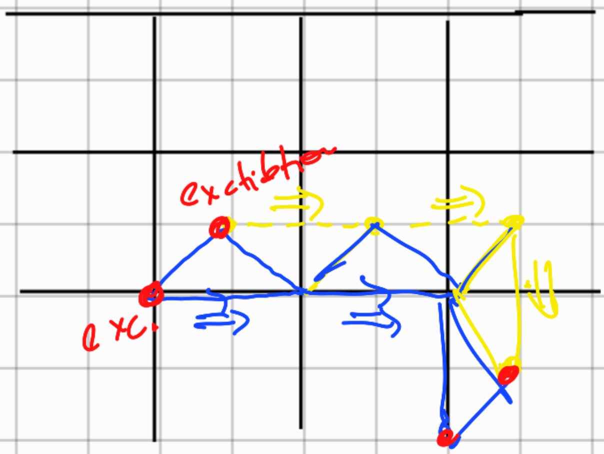
$$= \{e, C(123)\} \leftarrow C(e) \cup C(123)$$

$$\Rightarrow D \otimes D = (C(e), \square) \oplus (C(123), 0)$$

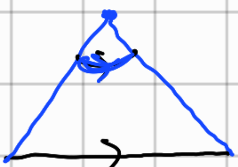
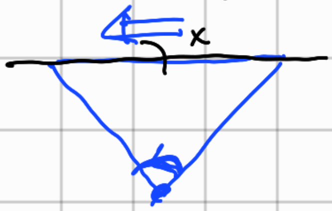
$\square = [+]_{\mathbb{Z}_2} \otimes [+]_{\mathbb{Z}_2}$, but this must be a irrep of S_3

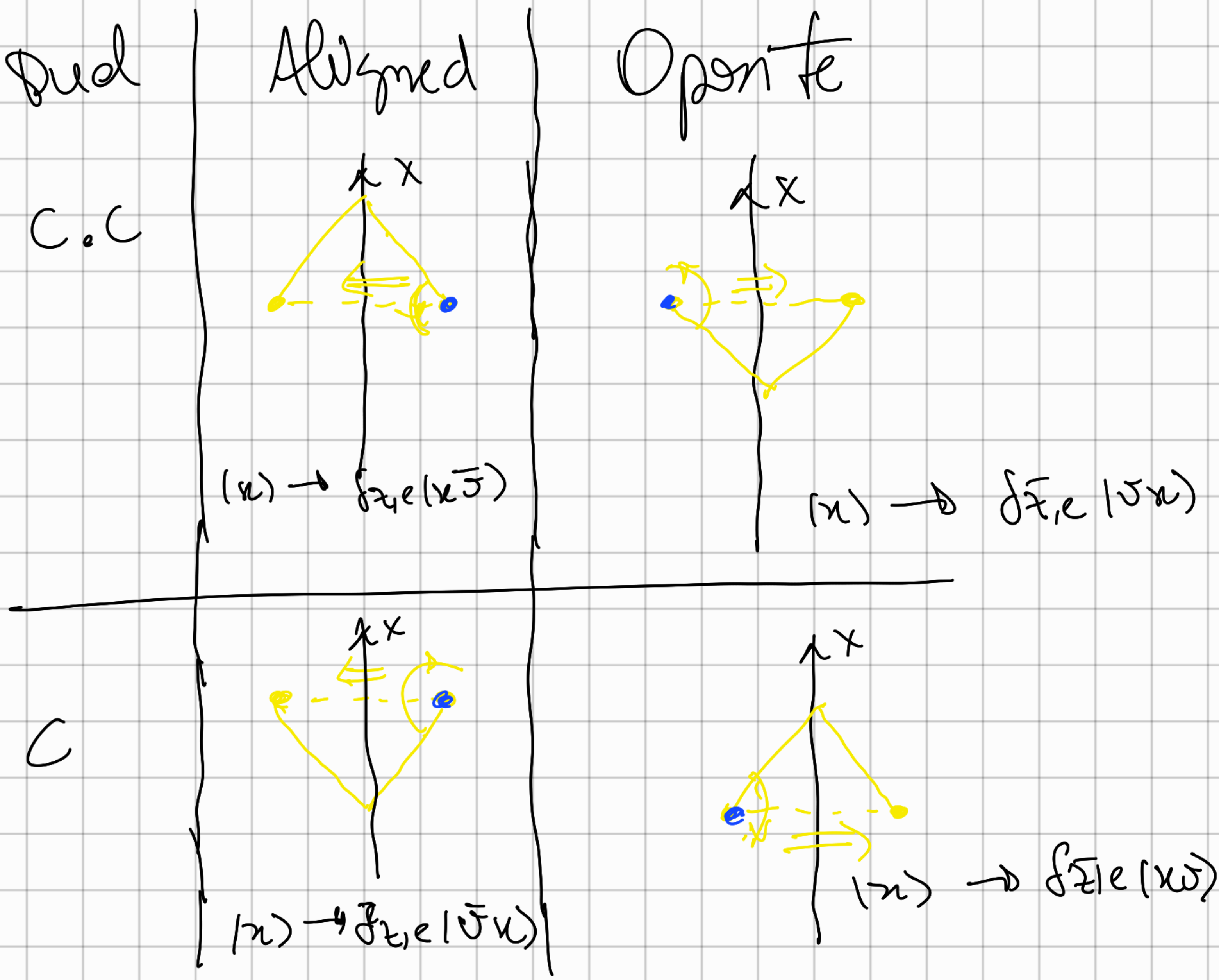
$\Rightarrow$ We need use the Induced representation.

Ribbon Operators (Anologous to line op. in Toric code)



- Building blocks (triangle operators)

original	Aligned	Opposite
C.C	$\|x\rangle \longrightarrow \delta_{z,x}\|x\rangle$  aligned	$\|x\rangle \longrightarrow \delta_{\bar{z},x}\|x\rangle$  x
C	 $\|x\rangle \longrightarrow \delta_{z,x}\|x\rangle$	 $\|x\rangle \longrightarrow \delta_{\bar{z},x}\|x\rangle$



• Two label: z, v.

the fusion operator is the smallest Ribbon operator

• We define the action of a Ribbon operator in vacuum by:

$$F^{(z,v)}|vac\rangle = F^{(z,\, zw\bar{z})}|vac\rangle = |z,w\rangle$$

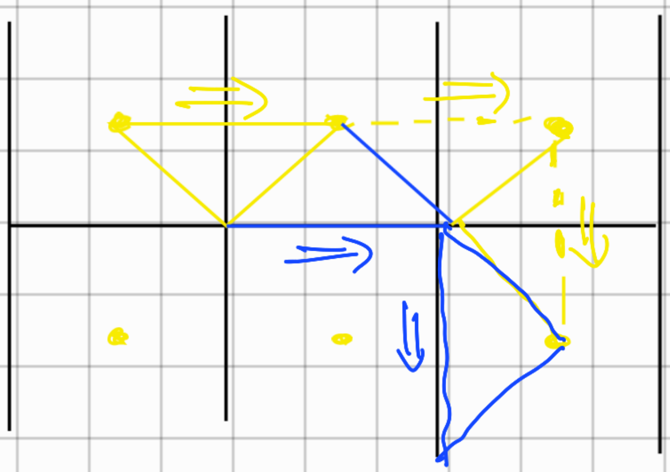
The Ribbon operator is defined by the recursive relation:

$$F^{(z,\sigma)}_{(\rho)} = \sum_{k\in G} F^{(k,\sigma)}_{(\rho_1)} F^{(\bar{k}z,\bar{k}\sigma k)}_{(\rho_2)}$$

where ρ is a ribbon composed of multiple triangles, such that $\rho = \rho_1 \cup \rho_2$.

$\longrightarrow$ This does not depend of how we decompose the ribbon

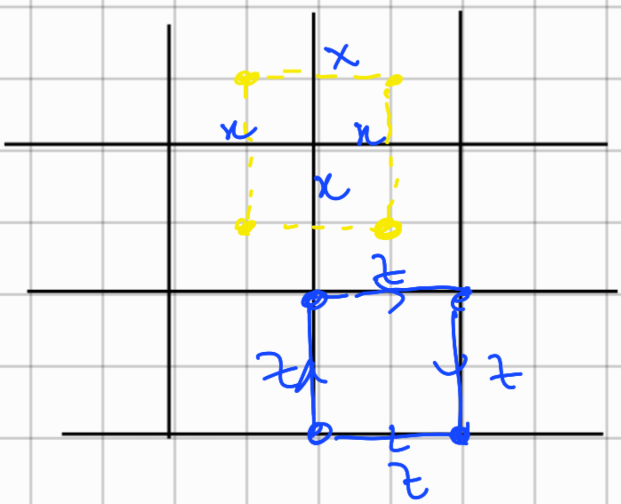
Simple example:



The Star and Plaquette operators as a closed Ribbon operator

Analogues in Toric Code

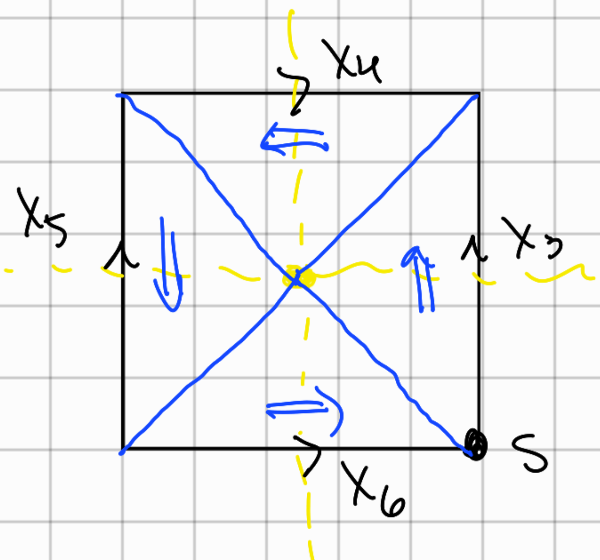
Toric Code !



A_s: smallest loop in dual lattice

B_p: smallest loop in original lattice.

- Consider the following plaquette



We aply the Ribbon Op $F^{(z,0)}_{(\tau)}$, as all triangle that compose the ribbon is original this Ribbon does not depend on v!

$$F^{(z)}(\tau) \quad , \text{with } \tau = \tau_3 \cup \tau_4 \cup \tau_5 \cup \tau_6$$
$$= \tau_{34} \cup \tau_{56}$$

$$F^{(z)}(\tau) = \sum_{k \in G} F^{(k)}(\tau_{34})\, F^{(\bar{k} z)}(\tau_{56}) =$$

$$= \sum_{k} \sum_{k', k''} F^{(k')}(\tau_3)\, F^{(\bar{k}' k)}(\tau_4)\, F^{(k'')}(\tau_5)\, F^{(\bar{k}'' \bar{k} z)}(\tau_6)$$

All the triangle operators are counterclockwise

and $\tau_3, \tau_6 \to$ aligned $(\delta_{z,x})$
$\tau_4, \tau_5 \to$ opposite $(\delta_{\bar{z},x})$

$$\Rightarrow F^{(z)}(\tau) = \sum_{k, k', k''} \delta_{k', x_3}\, \delta_{\overline{(\bar{k}' k)}, x_4}\, \delta_{\bar{k}'', x_5}\, \delta_{\bar{k}'' \bar{k} z, x_6}$$

$$= \sum_{k, k'} \delta_{k', x_3}\, \delta_{\bar{k} k', x_4}\, \delta_{x_5 \bar{k} z, x_6} =$$

$$= \sum_{k} \delta_{\bar{k} x_3, x_4}\, \delta_{x_5 \bar{k} z, x_6} = \sum_{k} \delta_{\bar{k}, x_4 \bar{x}_3}\, \delta_{x_5 \bar{k} z, x_6}$$

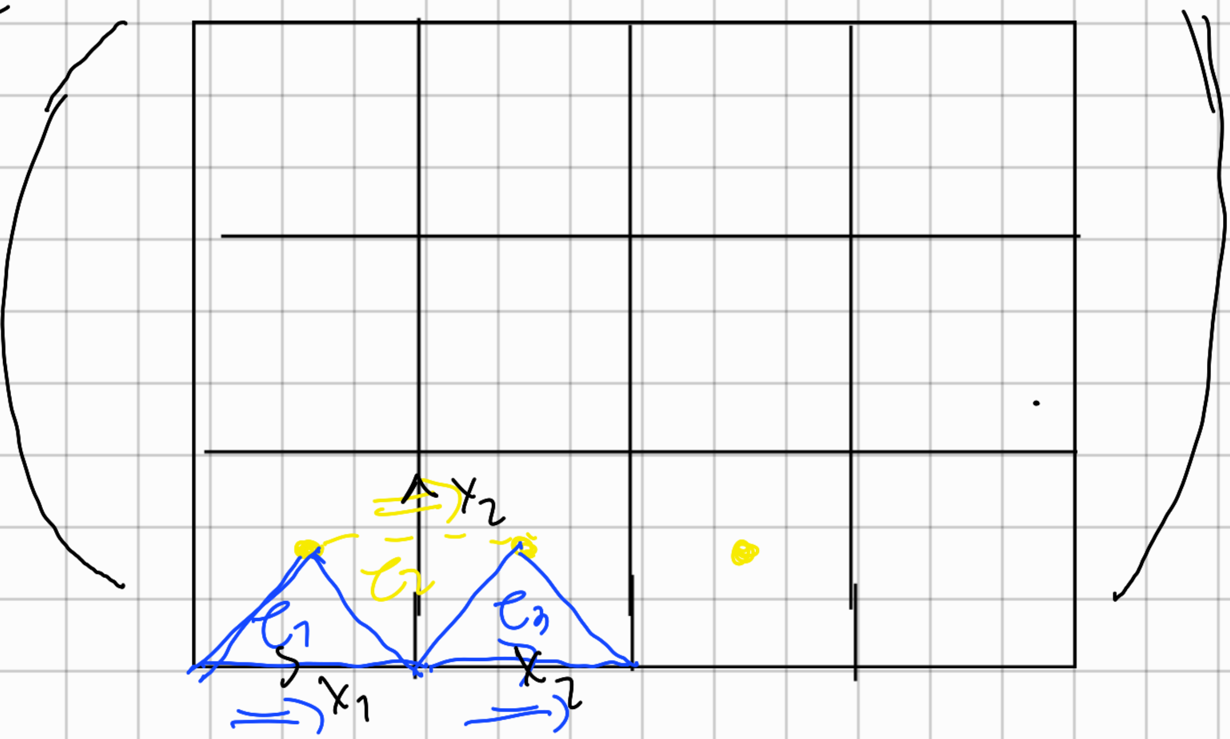
$$= \delta_{x_5 x_4 \bar{x}_3 z, x_6} = \delta_{x_4 \bar{x}_3 z, \bar{x}_5 x_6} = \delta_{\bar{x}_3 z, \bar{x}_4 \bar{x}_5 x_6}$$

$$= \delta_{z, x_3 \bar{x}_4 \bar{x}_5 x_6}$$

$$= B_p^z$$

Home work: Show that the smallest loop in the dual lattice correspond to A_s^g.

Ex:

$F^{(z,\sigma)}_{(\rho)}$ 

τ_1, τ_3 aligned counterclockwise.
τ_2 oposite counterclockwise.

$$\Rightarrow F^{(z,\sigma)}_{(\rho)}|x\rangle = \sum_{k} F^{(k,\sigma)}_{(\tau_1)} F^{(\bar{k}z,\bar{k}\sigma k)}_{(\tau_{23})} =$$

$$= \sum_{k,k'} F^{(k,\sigma)}_{(\tau_1)} F^{(k',\bar{k}\sigma k)}_{(\tau_2)} F^{(\bar{k}'\bar{k}z,\bar{k}'\bar{k}\sigma k k')}_{(\tau_3)} |x\rangle$$

$$= \sum_{k,k'} F^{(k,\sigma)}_{(\tau_1)} F^{(k',\bar{k}\sigma k)}_{(\tau_2)} \delta_{\bar{k}'\bar{k}z, x_3} |x\rangle$$

$$= \sum_{\kappa,\kappa'} F^{(\kappa,\vartheta)}(\tau_1)\, \delta_{\kappa',e}\, \delta_{\bar{\kappa}'\bar{\kappa}z,x_3} |\bar{\kappa}\vartheta\kappa x\rangle$$

$$= \sum_{\kappa,\kappa'} \delta_{\kappa,x_1}\, \delta_{\kappa',e}\, \delta_{\bar{\kappa}'\bar{\kappa}z,x_3} |\bar{\kappa}\vartheta\kappa x\rangle$$

$$= \sum_{\kappa} \delta_{\kappa,x_1}\, \delta_{\bar{\kappa}z,x_3} |\bar{\kappa}\vartheta\kappa x\rangle =$$

$$= \delta_{\bar{x}_1 z, x_3} |\bar{x}_1\vartheta x_1 x\rangle = \delta_{z,x_1x_3} L_+^{\bar{x}_1\vartheta x_1} |x\rangle .$$

$$\Rightarrow \boxed{F^{(z,\vartheta)}(\rho) = \delta_{z,x_1x_3} L_+^{\bar{x}_1\vartheta x_1}} ,$$

with $L_+^{\vartheta}|x\rangle = |\vartheta x\rangle$.

$\hookrightarrow$ same rotation that Kitaev use in his paper!

Homework: Consider $G = \mathbb{Z}_2$,

- Define the operators $L_+^h, L_-^h, T_+^z, T_-^z$.

$$L_+^h|x\rangle = |hx\rangle , \quad L_-^h|x\rangle = |x\bar{h}\rangle$$

$$T_+^z|x\rangle = \delta_{z,x}|x\rangle , \quad T_-^z|x\rangle = \delta_{\bar{z},x}|x\rangle$$

For $G = \mathbb{Z}_2 = \{e, g\}$, $g^2 = e$ $\quad L^g_+ = L^g_- = L^g$.

$T^z_+ = T^z_- = T^z$.

Each edge there is a Hilbert space

$\mathcal{H}_i = \mathrm{span}\big|_{\mathbb{C}} \{|e\rangle, |g\rangle\}$ (qubit)

a) Show ~~that~~ L^g correspond to spin flip op (σ^x).

b) Show that $T^e + T^g = 1$, and write the σ^z operator.

c) Using the triangle operator, show that $L^g_{\tilde{e}} = F^{(e,g)}_{(\tilde{e})}$, and
↳ dual lattice

$$T^e_e - T^g_e = F^{(e,e)}_{(e)} - F^{(g,e)}_{(e)}$$

And finally

$$W(\bar{\gamma}) = \prod_{e \in \bar{\gamma}} L^g_e = \prod_{e \in \bar{\gamma}} F^{(e,g)}_{(\tilde{e})},$$

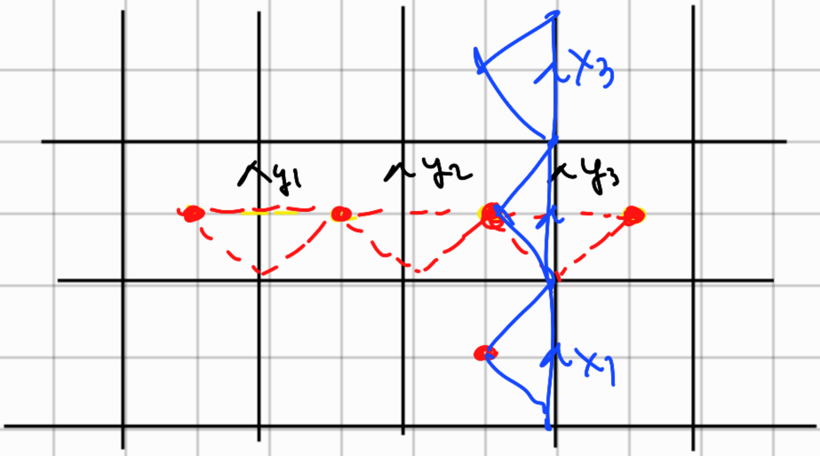
$$T(\gamma) = \prod_{e\in\gamma} \left(T_e^e - T_e^q \right) = \prod_{e\in\gamma} \left(F^{(e,e)}_{(e)} - F^{(q,e)}_{(e)} \right)$$

Satisfy

$$W(\bar{\gamma})\,T(\gamma) = -\,T(\gamma)\,W(\bar{\gamma}),$$

which we consider ~~the~~ the γ and $\bar{\gamma}$ lines touch just one time.

See image below:



$x_2 = y_3$!

Hint: show the algebra (commutation) relation for op $L^q_e\,T^a_e$.

Modular Data (S and T matrix)

Given two anyons (C_a, π), $(C_{a'}, \pi')$ the S matrix is given by:

$$S_{(C_a,\pi)(C_{a'},\pi')} = \frac{1}{|Z(a)||Z(a')|} \times$$

$$\times \sum_{h : hah^{-1} \in Z(a)} tr_{\pi}(ha'h^{-1})\, tr_{\pi'}(hah^{-1})$$

We can show ~~that~~ we can write as:

$$S_{(C,\pi)(C',\pi')} = \frac{1}{|G|} \sum_{\substack{g \in C' \\ h \in C \\ hg = gh}} \chi_{\pi}(g)\, \chi_{\pi'}(h) \quad , \quad \chi_{\pi}(g) = tr_{\pi}(g), \; g \in C'$$

$$\Rightarrow \chi_{\pi}(g) = tr_{\pi}(hgh^{-1})$$ for some h.

- We must be carefull since π cannot be a irrep of the whole group (we see this in fusion)

The S matrix can be used to compute the fusion coeficients

$$N_{xy}^{z} = \sum_{u \in L} \frac{S_{xu} S_{yu} S_{zu}^{*}}{S_{ou}}, \text{ where}$$

$L = \{$all irreps of $D(G)\}$ = Label set!

and $0 = (e, 1)$ (vacuum).

Consider G abelian:

Consider two anyon of $D(G)$

$\alpha = (\{a\}, \pi)$, $\beta = (\{b\}, \pi')$

then $$S_{\alpha,\beta} = \frac{1}{|G|} \sum_{\substack{g \in C_\beta \\ h \in C_\alpha \\ gh = hg}} \chi_\pi(g) \chi_{\pi'}(h)$$

$$\Rightarrow S_{\alpha,\beta} = \frac{1}{|G|} \sum_{\substack{g = b \\ h = a}} \chi_\pi(g) \chi_{\pi'}(h) \Rightarrow$$

$$\Rightarrow S_{\alpha,\beta} = \frac{1}{|G|} \chi_\pi(b) \chi_{\pi'}(h),$$

Since π is a irrep of $Z(a) = G$

then $\chi_\pi(b)$ exist. (same for $\chi_{\pi'}(a)$)

- Now consider α a vacuum

then $\alpha = (\{e\}, t)$, t: trivial irrep.

$$S_{\alpha\beta} = \frac{1}{|G|} \chi_\pi(b) \chi_\pi(e) = \frac{\chi_t(b)}{|G|}.$$

Since $\chi_t(b) = 1$ (trivial irrep)

$$S_{\alpha\beta} = \frac{1}{|G|} = \frac{d_\alpha d_\beta}{|G|} = \frac{1}{D},$$

D is the total quantum dimension $D := \sqrt{\sum_a d_a^2}$, for any G

$$D = |G|$$

Homework: Show that for any G

$$D = \sqrt{\sum_a d_a^2} = |G|$$

- Consider $G = \mathbb{Z}_n$, and two anyons with trivial flux (i.e., purely charge or vacuum)

I will use the $\mathbb{Z}_n = (\{0, 1, \dots, n-1\}, + \bmod n)$

two anyons (trivial flux)

$\alpha = (\{0\}, \chi_a)$ $\quad \chi_{a,b}$ irrep of $\mathbb{Z}_n$.

$\beta = (\{0\}, \chi_b)$ $\quad \chi_a(g) = e^{\frac{2\pi i a g}{n}}$.

$$S_{\alpha\beta} = \frac{1}{D} \sum_{\substack{g \in C_b \\ h \in C_a}} \chi_a(g) \chi_b(h) =$$

$$= \frac{1}{D} \chi_a(0) \chi_b(0) = \frac{d_a d_b}{D},$$

$d_a = 1 \ \forall a$ when G is abelian!

Therefore two trivial flux has trivial braidings.

* Now consider two anyons purely magnetic, i.e.,

$$\alpha = (\{a\}, \chi_0) \quad , \beta = (\{b\}, \chi_0)$$

$$S_{\alpha\beta} = \frac{1}{D} \sum_{\substack{g \in C_b \\ h \in C_a}} \chi_0(g)\chi_0(h) =$$

$$= \frac{1}{D} \chi_0(b)\chi_0(a) = \frac{d_a d_b}{D}$$

therefore two trivial charge anyons has braiding trivial.

Then we can define

$$\mathcal{L}_{0_e} = \{ \underbrace{(0,0)}_{\text{vacuum}}, (1,0), (2,0), \ldots, (n-1,0) \}$$

All this anyons in this set has braiding trivial and the fusion is closed in the set.

$\mathcal{L}_m = \{(0,0), (0,1), \ldots, (0,n-1)\}$,

same as $\mathcal{L}_e$.

<u>Question</u>: $\exists$ a dyon ~~that~~ can have braiding trivial with an purely electric anyon?

Answer: Yes!

Consider a anyon with trivial flux but with "electric" charge

$$a = (0, \chi_a), \quad \chi_a(g) = e^{\frac{2\pi i}{n} a g}$$

and a dyon:

$$\beta = (b, \chi_b), \quad b \neq 0.$$

$$S_{\alpha,\beta} = \frac{1}{D} \sum_{\substack{g=b \\ h=0}} \chi_a(g) \chi_b(h) \Rightarrow$$

$$\Rightarrow S_{\alpha,\beta} = \frac{1}{D} \chi_a(b) \underbrace{\chi_b(0)}_{=1} \Rightarrow$$

$$S_{\alpha,\beta} = \frac{1}{D} e^{\frac{2\pi i ab}{n}},$$

if $\exp\left(\frac{2\pi i ab}{n}\right) \neq 1$ then

α, β has non trivial braiding.

this is true if ab is not multiple of n, i.e,

$ab = kn$ (braiding trivial)

$k \in \mathbb{N}$.

$$\Rightarrow \frac{ab}{k} = n \quad a, b \in \{1, 2, \dots, n-1\},$$

if we take n prime number then ab is not a multiple of n.

then for $D(\mathbb{Z}_n)$ with n non prime L_{oe}, L_{om} define the Lagrangian subalgebra.

→ For n prime we need take careful!

<u>Exemple</u>: $G = \mathbb{Z}_2 = \{1, m\}$. prime

anyons: $1, e, m, f$.

$L_{oe} = \{1, e\}$, $L_{om} = \{1, m\}$

$e = (0, \chi_1)$, $m = (1, \chi_0)$

$$S_{em} = \frac{1}{D} \chi_1(1) \chi_0(0) = \frac{e^{\frac{2\pi i}{2}}}{D} =$$

$$\Rightarrow S_{em} = \frac{1}{2} \cdot e^{i\pi} = -\frac{1}{2} \neq \frac{d_e d_m}{2}$$

$\Rightarrow$ non trivial braiding

$G = \mathbb{Z}_3 = \{0, 1, 2\}$ prime

Anyons :

vacuum	electric	magnetic	dyon
$(0,0)$	$(0,\chi_1)$	$(1,\chi_0)$	$(1,\chi_1)$
	$(0,\chi_2)$	$(2,\chi_0)$	$(1,\chi_2)$
			$(2,\chi_1)$
			$(2,\chi_2)$

$D = 3$.

$$S_{(2,\chi_0)(2,\chi_1)} = \frac{1}{D} \sum_{\substack{g=2 \\ h=2}} \chi_0(g)\chi_1(h)$$

$$= \frac{1}{D} \chi_0(2)\chi_1(2) = \frac{1}{D}\chi_1(2) =$$

$$= \frac{e^{\frac{2\pi i \cdot 2}{3}}}{D} = e^{\frac{4\pi i}{3}}$$ (non trivial phase)

Consider $G = \mathbb{Z}_4$, $(4 = 2 \cdot 2.)$

Anyons

Vacuum	electric	magnetic	dyon
(0,0)	(0,1)	(1,0)	(1,1)
	(0,2)	(2,0)	(1,2)
	(0,3)	(3,0)	(1,3)
			(2,1)
			(2,2)
			(2,3)
			(3,1)
			(3,2)
			(3,3)

$D = 4.$

$\# \text{anyon} = 4^2 = 16.$

take the anyon $\alpha = (2,0)$

$\beta = (2,2)$

$$S_{\alpha,\beta} = \frac{1}{D} \sum_{\substack{g=2 \\ h=2}} \chi_0(g)\, \chi_2(h) =$$

$$\Rightarrow S_{\alpha,\beta} = \frac{1}{D} \chi_0(2)\chi_2(2) =$$

$$= \frac{1}{D} e^{\frac{2\pi i \cdot \cancel{2} \cdot 2}{\cancel{4}}} = \frac{d_\alpha d_\beta}{D}.$$

they braiding trivially!

But to introduce they in $\mathcal{L}_{oe}$,
we need that the fusion is closed
in the set, i.e, $\otimes$ must satisfy

$$\otimes\ \mathcal{L}_{oe} \times \mathcal{L}_{oe} \longrightarrow \mathcal{L}_{oe}.$$

in this case

$\forall$ anyon $\in$ in "electric" class
they are $(0,a)$ $a \in \{1,2,3\}$
and the dyon has 2 flux:

$\beta = (2,2)$,

$$\underbrace{\alpha \otimes \beta}_{\notin \text{ in electric}} = (2+0, a+2) = (2, a+2)$$

we need to
include another anyon!

maybe we need to introduce all 2-flux anyons, they all braiding trivially, and now the set

$\mathcal{L}_{oe} = \{(0,0), (0,1), (0,2), (0,3),$
$, (2,1), (2,2), (2,3)\}$

Now I think! (homework)

they all braiding trivially but the fusion is not closed, e.g.,

$$(0,3) \otimes (2,1) = (2, 3+1 \bmod 4) = \\ = (2,0) \in \mathcal{L}_m,$$

$(2,0)$ does has non trivial braiding with electric anyons.

- The case with non Abelian anyon is more interesting!

· For example we know ~~that~~ for $D(S_3)$ the anyons D and F are purely electric, i.e., $D = (C(12), +)$, $F(C(123), +)$

then we expect them to have trivial braiding.

Lets check:

$$S_{D,F} = \frac{1}{D} \sum_{\substack{g \in C_F \\ h \in C_D \\ : gh = hg}} \chi_+(g) \chi_+(h),$$

But $\nexists\, gh = hg$:

$g \in \{(123), (132)\}$

$h \in \{(12), (13), (23)\}$.

therefore

$S_{D,F} = 0.$, and we can conclude that any anyon in $C(12)$ class has non trivial braiding with anyons in $C(123)$ class, i.e.,

$$S_{DF} = S_{DG} = S_{DH} = 0$$

Then we can consider the Lagrangian sub algebra $\mathcal{L}_{e} = \{A, D\}$, or $\mathcal{L}_{e} = \{A, F\}$?

First $\exists$ "magnetic" anyon which braiding trivially with a "electric" anyon?

<u>Answer: Yes!</u>

Consider B and F anyon,

$B = (\{e\}, -)$, $F = (C(123), +)$

$$S_{BF} = \frac{1}{6} \sum_{\substack{g \in \{(123), (132)\} \\ h = e}} \chi_-(g) \underbrace{\chi_+(h)}_{1} =$$

$$= \frac{1}{6} \sum_{g \in \{(123), (132)\}} \chi_-(g) = \frac{2}{6} = \frac{d_B d_F}{D}.$$

(B, F): Braiding trivially!

<u>Fusion:</u> $B \otimes F = F$.

Then $\mathcal{L} = \{A, B, F\}$ forms a lagrangian algebra,

with an "magnetic" anyon and a "electric" anyon!

Homework: Check if there are another anyons in $D(S_3)$ ~~that~~ braiding trivially with B and F.

T matrix

T matrix is diagonal given by:

$$T_{(c,\rho)(c',\rho')} = \delta_{c,c'} \delta_{\rho\rho'} \theta_{(c,\rho)},$$

where $\theta_{(c,\rho)} = \dfrac{\chi_\rho(g)}{\chi_\rho(e)} = \dfrac{\chi_\rho(g)}{\dim\rho}$,

$g \in C$, and $\chi_\rho(g)$: character of irrep ρ of $Z(g)$.

Ex: Consider $G = \mathbb{Z}_n := \{0, 1, \dots, n-1\}$ (+ mod n)

then $\alpha \in D(G) \Rightarrow \alpha = (a, b)$, $a, b \in \{0, 1, \dots, n-1\}$.

$$\theta_\alpha = \frac{\chi_\rho(g)}{\dim\rho}, \quad \dim\rho = 1 \Rightarrow$$

$$\Rightarrow \theta_\alpha = \chi_b(a) = e^{\frac{2\pi i b\cdot a}{n}},$$

$\theta_\alpha = 1$ (boson) if $b\cdot a = n \pmod n$

$\theta_\alpha = -1$ (fermion) if $b\cdot a = \frac{n}{2} \pmod n$

Then if n is odd, $\Rightarrow$ $\nexists$ fermion in $D(\mathbb{Z}_n)$, i.e.,

All anyons are bosons or anyon (θ is a phase)

- Now consider α a "magnetic" anyon

then $\theta_\alpha = e^{\frac{2\pi i a \cdot b}{n}} = 1$ (since $a = 0$)

$\forall \alpha \in \mathcal{L}_m$.

- Consider α a "electric" anyon

then $\theta_\alpha = e^{\frac{2\pi i a \cdot b}{n}} = 1$ (since $b = 0$)

$\forall \alpha \in \mathcal{L}_e$.

- Consider a generic G, $\alpha \in D(G)$, with

α is "electric" anyon: $\alpha = (C_a, \tau)$

$\theta_\alpha = \frac{\chi_\tau(g)}{1} = 1$, (electric $\to$ boson)

α is "magnetic" anyon, $\alpha = (\{e\}, \rho)$

$\theta_\alpha = \frac{\chi_\rho(e)}{\dim \rho} = \frac{\dim \rho}{\dim \rho} = 1$ (boson).

Therefore electric (magnetic) anyons are bosons.

Homework Consider $\alpha \in D(S_3)$, construct the T matrix for $D(S_3)$.

The $SL(2,\mathbb{Z})$

Matrix S, and T is a representation of $SL(2,\mathbb{Z})$, they satisfy

$$S^4 = 1, \quad (ST)^3 = S^2, \text{ in general}$$

$$(ST)^3 = e^{\frac{2\pi i c_-}{8}} S^2,$$

for $D(G)$ $c_- = 0$

$\cancel{\exists}$ chiral modes!