Lecture 12

Topological Quantum Field theory

- CFTs
- Algebraic QFT
- TQFTs (80's) ✓

TQFTs are maps between categories
↳ functors

Recap:

Def: Categories $\mathcal{C}$ are specified by

- objects $\mathrm{obj}(\mathcal{C})$
- Morphisms $\mathrm{Hom}(X,Y)$, $X, Y \in \mathrm{Obj}(\mathcal{C})$

i) $\forall X \in \mathrm{Obj}(\mathcal{C})$, $\exists$ $id_X \in \mathrm{Hom}(X,X)$

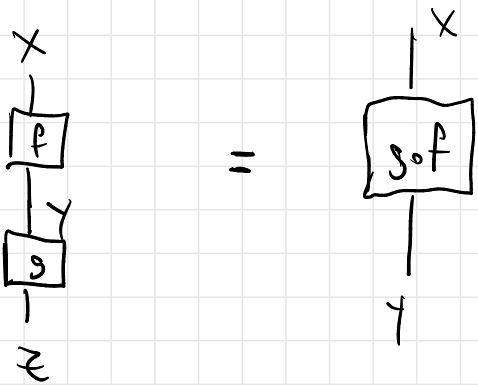
diagrammatically:

$$\begin{tikzcd} X \arrow[d, dash, "id_X"] \\ X \end{tikzcd}$$

ii) Composition of morphism

$f \in \mathrm{Hom}(X, Y)$, $g \in \mathrm{Hom}(Y, Z)$

we can compose $g \circ f \in \mathrm{Hom}(X, Z)$



$$[g \circ f](X) = g(\overbrace{f(X)}^{Y}) = Z$$

<u>Def</u>: Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.
A functor is a map

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

that assigns an object

$X \in \mathcal{C}$

into

$F(X) \in \mathrm{Obj}(\mathcal{D})$

$\forall\ X \in \mathcal{C}$

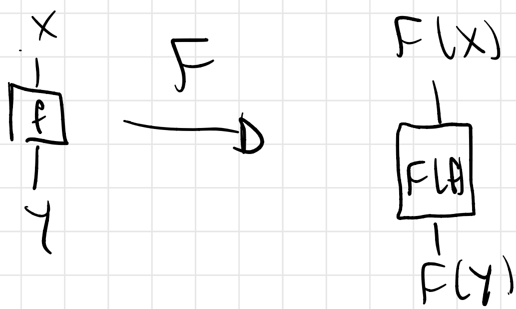
Stacking

$\boxtimes : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$

is a functor.

and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$

into $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$



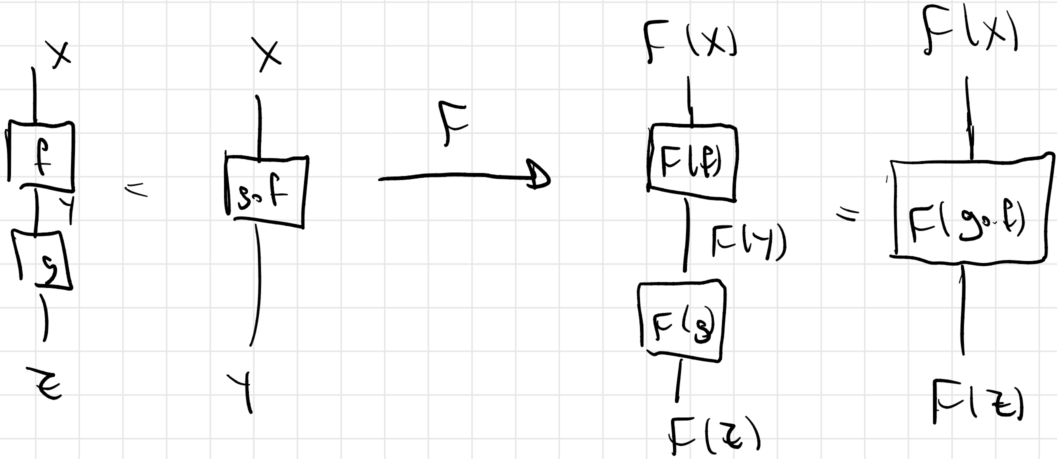
such that

i) Identity morphisms are preserved

$$F(id_X) = id_{F(X)} \in \mathrm{Hom}(F(X), F(X))$$

ii) F respects composition of morphisms.

$$F\left(g \circ_{\mathcal{C}} f\right) = F(g) \circ_{\mathcal{D}} F(f)$$



Notation: $\begin{cases} X \in \mathrm{Obj}(\mathcal{C}) \\ f \in \mathrm{Hom}(X, Y) \end{cases} \Longrightarrow \begin{cases} X \in \mathcal{C} \\ f \in \mathcal{C} \end{cases}$

As an example, consider

$$\mathcal{C} = \mathrm{Vec} \begin{cases} \mathrm{Obj}(\mathrm{Vec}) \ni X \text{ are finite-dimensional vector spaces over } \mathbb{C} \\ \mathrm{Hom}(X,Y) \ni f \text{ are linear maps } f : X \to Y \end{cases}$$

composition of morphisms = comp. of linear maps.

Vec on $\mathcal{C} = \{1\}$, $|X| = 1$ $\quad X = 1 \oplus 1 \oplus \dots \oplus 1$.

and

$D = \mathrm{Set}$ $\begin{cases} \mathrm{Obj}(\mathrm{Set}) \ni X \sim \text{sets} \\ \mathrm{Hom}(X,Y) \ni f \sim \text{functions between sets} \\ \text{comp. of morphisms} = \text{comp. of functions} \end{cases}$

Forgetful functor

$$F : \mathrm{Vec} \longrightarrow \mathrm{Set}$$

that forgets the linear structure

$\begin{cases} \text{addition} \\ \text{scalar multiplication} \\ \text{linearity of morphisms} \end{cases}$

$F(X) \sim$ set underlying $X \in \mathrm{Vec}$.

$F(f) \sim$ function defined by f, ignoring linear requirements.

Bordism category (cobordism)

Let us consider oriented smooth manifolds of dimension $d-1$:

$$\Sigma_{in} \quad \text{and} \quad \Sigma_{out}$$

$(d-1)$ manifold

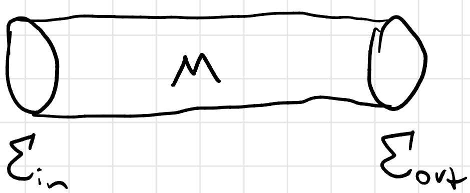
A cobordism M between Σ_{in} and Σ_{out} is a smooth d-dim manifold with boundaries

$$\partial M = \Sigma_{in} \sqcup \Sigma_{out}$$

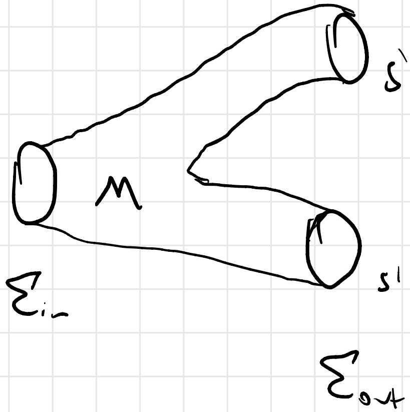
↳ disjoint union.

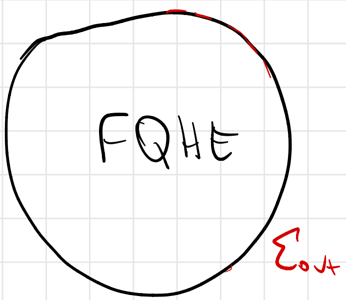
Example $(d=2)$

- Σ_{in} and Σ_{out} are S^1. A cylinder is a cobordism

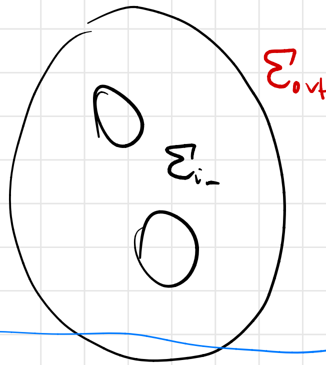


- Pair of pants $\Sigma_{in} = S^1$, $\Sigma_{out} = S^1 \times S^1$



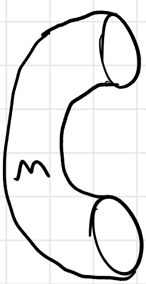


with two holes



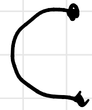
Example $\Sigma_{in} = \emptyset$ and $\Sigma_{out} = S^1 \times S^1$

$\Sigma_{in} = \emptyset$



$\Sigma_{out} = S^1 \times S^1$

• Example $d = 1$

Σ_{in}  Σ_{out} , $\Sigma_{in} = \emptyset$  Σ_{out}

M

d-dim cobordism form a category!

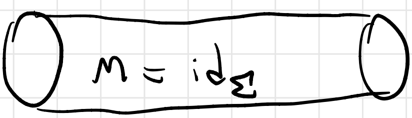
Bord_d

with:

• $\mathrm{Obj}(\mathrm{Bord}_d) \ni \Sigma$ are $d-1$ dimensional orientable manifolds

• $\mathrm{Hom}(\Sigma_{in}, \Sigma_{out}) \ni f$ are d dimensional bordisms between Σ_{in} and Σ_{out}

i) Identity morphism $\mathrm{id}_\Sigma \in \mathrm{Hom}(\Sigma, \Sigma)$

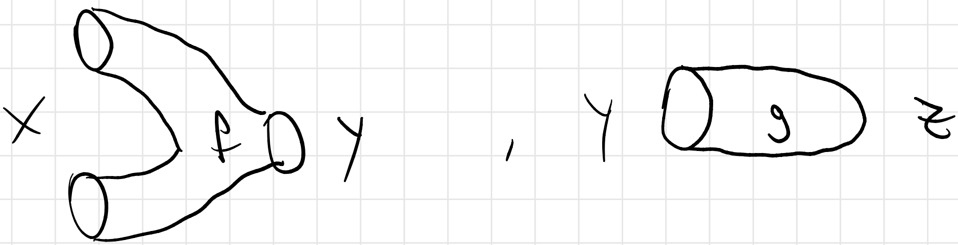
Σ 

Σ

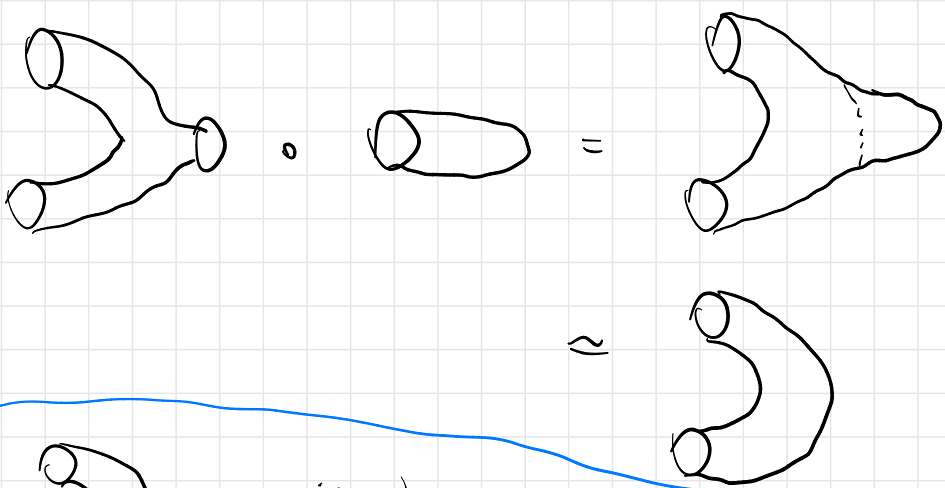
cylinder $M = \Sigma \times [0,1]$.

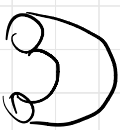
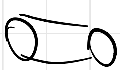
ii) Composition of morphisms come from gluing cobordism along shared boundaries

$X = S^1 \times S^1$, $Y = S^1$, $Z = \emptyset$



Composition:



Aside:  $\xrightarrow{\text{isomorphism}}$  $\in \mathrm{Hom}(Y, Z)$.

$\mathrm{Hom}(S^1 \times S^1, \emptyset)$ $\quad$ $\mathrm{Hom}(S^1, S^1)$

$|\psi\rangle$ $\langle\psi|$

$g_{\mu\nu}$ $g^{\mu\nu}$

TQFT (Atiyah - Segal 1988)

Def:

A d-dimensional TQFT is a symmetric monoidal functor

$$\boxed{Z : Bord_d \longrightarrow Vec}$$

↳ Z translates topological data into algebraic data.