Lecture 13

• Recap: categories, functors, Vec, Bord_d

Def: TQFTs

A d-dimensional TQFT is a symmetric monoidal functor

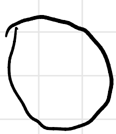
$$\boxed{Z : \text{Bord}_d \longrightarrow \text{Vec}}$$

↳ Z translates topological spaces (manifolds + cobordisms) into algebraic data (vector spaces + linear maps)

↳ (Atiyah 1988): characterize topological invariants.

• Action on objects.

$\Sigma \in \text{Obj}(\text{Bord}_d)$ correspond to closed, oriented $(d-1)$-dimensional manifold.



$\xrightarrow{\ Z\ }$ $Z(\Sigma) \in \text{Obj}(\text{Vec})$

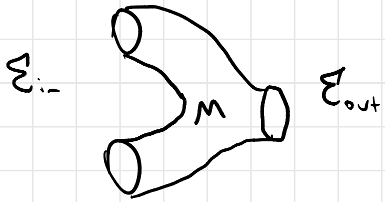
(finite-dim vector space)

$\Sigma = S^1$

Z assign a vector space to Σ.

• Action on morphisms

Morphisms in Bord_d are d-dimensional (oriented) cobordisms

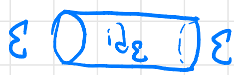


$$\xrightarrow{\;Z\;} \quad Z(M) : Z(\Sigma_{in}) \longrightarrow Z(\Sigma_{out})$$

"time evolution operator"

Functoriality:

i) Z preserves identity

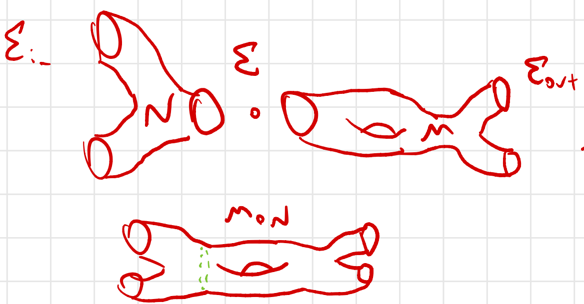


$$Z(id_\Sigma) = id_{Z(\Sigma)} \quad \text{(identity linear map)}$$

$$id_{Z(\Sigma)} : Z(\Sigma) \longrightarrow Z(\Sigma)$$
$$u \longmapsto u.$$

ii) Z preserves composition.

$$Z(M \circ N) = Z(M) \cdot Z(N)$$



$\longrightarrow$

$Z(M) \cdot Z(N) : Z(\Sigma_{in}) \to Z(\Sigma_{out})$

$Z(N) : Z(\Sigma_{in}) \to Z(\Sigma)$

$Z(M) : Z(\Sigma) \to Z(\Sigma_{out})$

↳ Gluing spacetimes ~ composing linear operators.

→ the two categories above have a monoidal structure

$$\Sigma_1, \Sigma_2 \in Bord_d \implies \Sigma_1 \sqcup \Sigma_2 \in Bord_d$$

disjoint union.

$$V_1, V_2 \in Vec \implies V_1 \otimes V_2 \in Vec$$

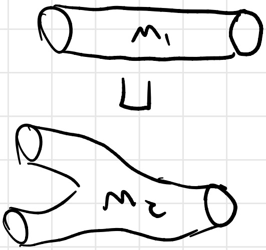
tensor product

TQFT is a monoidal functor:

$$Z(\Sigma_1 \sqcup \Sigma_2) \simeq Z(\Sigma_1) \otimes Z(\Sigma_2)$$

$$Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2)$$

↳ Independent systems factorize into tensor products (locality)



$\simeq$

$Z(M_1)$

$\otimes$

$Z(M_2)$

TQFT Z is symmetric:

$$Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$$
$$= Z(\Sigma_2) \otimes Z(\Sigma_1)$$

Let us note that for closed cobordisms

$\Sigma_{in} = \emptyset$

$\Sigma_{out} = \emptyset$

$$\underbrace{Z(\Sigma)}_{n\text{-dim}} = Z(\Sigma \sqcup \emptyset) = \underbrace{Z(\Sigma)}_{n\text{-dim}} \otimes Z(\emptyset)$$

$\Rightarrow$ $Z(\emptyset) \sim$ 1-dim vector space

$\Rightarrow$ $Z(\emptyset) \simeq \mathbb{C}$

$$Z(M) : Z(\emptyset) \longrightarrow Z(\emptyset)$$

$$Z(M) \approx \mathrm{End}(Z(\emptyset)) \simeq \mathbb{C}$$

$\Rightarrow$ $Z(M)$ is a c-number. (partition function).

Physics dictionary:

d-dim spacetime M

$$S_M[\phi] = \int_M d^d x \; L[\phi]$$

$$Z(M) = \int D\phi \; e^{i S_M[\phi]}$$

Math people: $Z(M)$ topological invariant of M

Physics people: properties of $Z(M)$

In physics, TQFT is a QFT that is metric independent

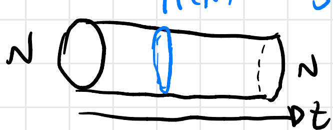
E.g: K-matrix Chern-Simons

$$L[a] = \frac{K_{IJ}}{4\pi} \epsilon^{\mu\nu\rho} a_\mu^{(I)} \partial_\nu a_\rho^{(J)}$$ ← Rieli: May 4^{th}

$$Z = \int [Da] \; e^{i \int_M L[a]}$$

$$\overset{d}{M} = \overset{d-1}{N} \times [0,1]$$

$\mathcal{H}(N) \sim$ ground state space underlying Hamiltonian



Turaev - Viro - Barrett - Westbury TQFTs

(TVBW TQFTs)

Input data | Output

- Manifold M_3
- Fusion category $\mathcal{C}$

$\mathcal{C} = (L, N_c^{ab}, \{F_d^{abc}\})$

$\longrightarrow$ (2+1)D unitary TQFT

Manifold M is implemented via triangulation (simplicial complexes)

$\Rightarrow$ Every compact, closed, orientable 3-manifold can be represented by simplicial triangulation

Simplicial complexes : capture properties of M_d into sets of <u>oriented</u>

0 - dim (points) $v \in V$

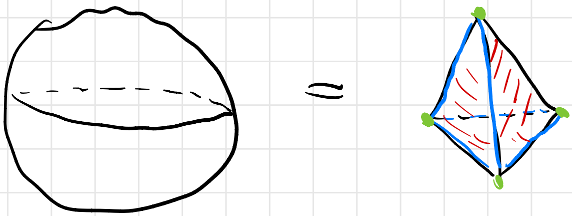
1 - dim (edges) $e \in E$

2 - dim (faces) $f \in F$

$\vdots$

d - dim (hypervolumes)

Ex: $d = 2$

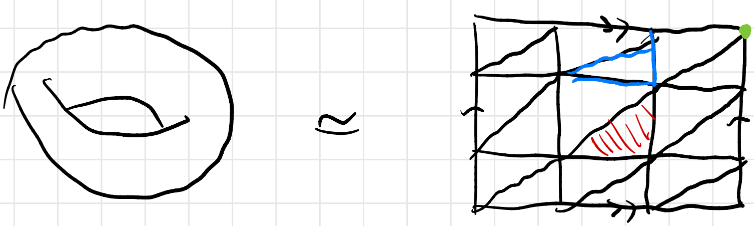


0 - cells

1 - cells

2 - cells

(no interior)



$\Rightarrow$ triangulations are not unique

$\rightsquigarrow$ two triangulations for M_d are equivalent if they can be deformed into each other through a Pachner's move

$\rightsquigarrow$ We want to make sure that $\chi(M)$ is triangulation independent

$\rightsquigarrow$ a true topological invariant of M.

Let us consider a fusion category

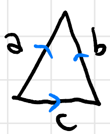
$$\mathcal{C} = (L, N_c^{ab}, \{F_d^{abc}\}).$$

We define a state to be the following:

<u>Def</u>: A (admissible) state

$$s: E \to L$$

is a coloring of the edges by simple objects, where fusion rules are enforced at faces



is admissible if $N_c^{ab} \neq 0$.

$$\xrightarrow{\quad a \quad} \;=\; \xleftarrow{\quad a^* \quad}$$

Def: <u>TVBW TQFT</u>

Input: • $\mathcal{C} = (L, N_c^{ab}, \{F_d^{abc}\})$

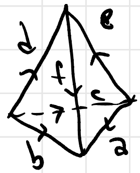
• Manifold M with triangulation (V, E, F, T)

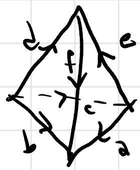
For each simplex in the triangulation:

$$\bullet \longmapsto D = \sqrt{\sum_{a \in L} d_a^2}$$

$$\xrightarrow{\quad a \quad} \longmapsto d_a \qquad\qquad d_a = d_{a^*}$$

 $\longmapsto \Theta(f) = \sqrt{d_a d_b d_c}$

$+$  $\longrightarrow \quad t^+ = \sqrt{d_a d_b d_e d_d}\ [F^{abd}_e]_{c,f}$

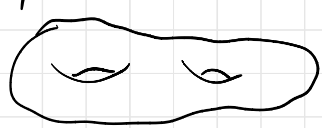
$-$  $\longrightarrow \quad t^- = \sqrt{d_a d_b d_e d_d}\ [F^{abc}_e]^{-1}_{f,c}$

We define the state-sum TVBW TQFT as

$$\boxed{Z_{\mathcal{C}}(M) = \frac{\displaystyle\sum_{S \text{ admissible}} \prod_{e\in E} d_{S(e)} \prod_{t\in T} t^{o(t)}}{\displaystyle\prod_{v\in V} D \prod_{f\in F} \Theta(f)}}$$

- triangulation independent, which follows from the pentagon equation for $\mathcal{C}$.
- Gauge invariant (F-symbols)

For the identity cobordism with boundary

$M = \Sigma \times [0,1]$, $\Sigma =$ 

$Z_{\mathcal{C}}(\Sigma)$ is a finite dimensional vector space.

$Z_{\mathcal{C}}(\Sigma) = \mathcal{H}_0^{LW}(\Sigma)$ on ground state space of Levin-Wen Hamiltonian.

TVBW TQFT corresponds to an effective field theory to Levin-Wen string net models)

$\rightsquigarrow$ $Z_{\mathcal{C}}(M)$ has emergent higher-form symmetries (lines)

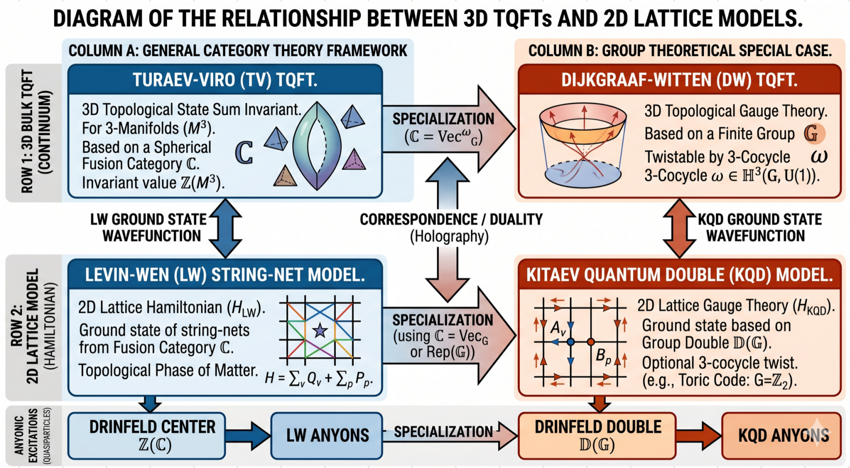
$\hookrightarrow$ Are captured by simple objects of $Z(\mathcal{C})$ Drinfeld center of $\mathcal{C}$!

$\rightsquigarrow$ World lines of anyons.

$$a \times b = \sum_c N_c^{ab}\, c \quad \longleftrightarrow \quad W_a W_b = \sum_c N_c^{ab}\, W_c$$

Emergent non-invertible 1-form symmetries!

Relationship between TVBW TQFTs, Levin-Wen string nets; and Dijkgraaf-Witten TQFTs and Kitaev Quantum Doubles



source: Gemini