K- matrix formulation

U(1) Chern- Simons

↳ Bulk

↳ Edge

$U(1)^N$ Chern - Si

↳ Bulk

↳ Edge

U(1) Chern- Simons

$$\rightarrow \quad S[a] = \int \frac{K}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \cdot \underline{\partial_\nu} a_\rho$$

→ TQFT

$K \rightarrow$ level $\quad K \in \mathbb{Z}$ (large gauge transf)

$a_\mu \rightarrow$ gauge emergent fields

Invariants are global quantities

Hamiltonian is zero

e.o.m $\quad f_{\mu\nu} = 0$ (flat connections)

→ CS term is gauge - invariant up to boundary terms

$$Q(\Sigma_1) = \int_{\Sigma_1} a \quad \rightarrow \text{gauge invariant}$$

$$W_n = \exp\left(in \oint_C a \right) \to \text{worldline of q.p.s.}$$

* Couplings

↳ EM field ⇝ Hall effect, filling fraction

$$S[a] = \int \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \cdot \underline{\partial_\nu} a_\rho - A_\mu J^\mu$$

$$J^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho$$

Integrate out a_μ

$$\text{e.o.m} \to a_\mu = \frac{1}{K} A_\mu \quad (\text{locally})$$

→ gauge invariant ?

CS term A_μ → level $1/K$ ⇒ $\nu = 1/K$

K odd → Laughlin states

$$\langle J^\mu{}_{ind} \rangle = - \frac{\delta S[A]}{\delta A_\mu}$$

$$J^i{}_{ind} = \frac{1}{2\pi K} \varepsilon^{ij} E_j$$

↳ Hall current (Hall effect)

* Matter current → charge-flux relation & qp charge

$$S[a] = \int \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \cdot \underline{\partial_\nu} a_\rho - a_\mu j^\mu$$

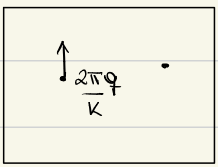
$$j^{\mu} = \frac{q}{2\pi} \varepsilon^{\mu\nu\rho} \partial_\nu b_\rho \rightarrow \text{is conserved}$$

e.o.m

$$j^{i} = \frac{K}{2\pi} \varepsilon^{ij} E_j$$

$$\boxed{\rho = j^{0} = \frac{K}{2\pi} B} \rightarrow \text{charge - flux}$$

$$\hookrightarrow \frac{2\pi q}{K}$$



$$S[a] = \int \frac{K}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho -$$

$$\frac{q}{2\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu b_\rho$$

Integrate out a_μ

$$\text{e.o.m} \Rightarrow a_\mu = -\frac{1}{K} A_\mu + \frac{q}{K} b_\mu$$

$$S = \int \left(-\frac{q}{K} A_\mu j^{\mu} \right)$$

$\underbrace{Q = -\frac{q}{K}}$ in general is fractional

$K = 1$, IQH $\Rightarrow Q = |q|$

* Quantização

$$[a_1(\vec{x}), a_2(\vec{y})] = \frac{2\pi i}{K} \delta^2(\vec{x}-\vec{y})$$

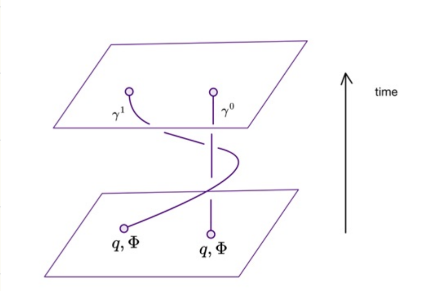
$$W_i = e^{i\int dx^i a_i}$$

$$W_1 W_2 = e^{-2\pi i/K} W_2 W_1$$

$$(W_1)^n (W_2)^m = e^{-2\pi i mn/K} W_2 W_1$$

$(W_1)^K \rightarrow$ transparent lines

$$(W_{\gamma_0})(W_{\gamma_1}) = e^{-2\pi i q^2/K \, \text{Link}(\gamma^0, \gamma^1)} W_{\gamma_1} W_{\gamma_0}$$



$$\theta = \frac{q^2}{K} \bmod 2$$

$GSD = K$

↳ Edge theory

$$S[a] = \int_M \frac{K}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$$

$\partial M \rightarrow$ boundary terms cannot be neglect

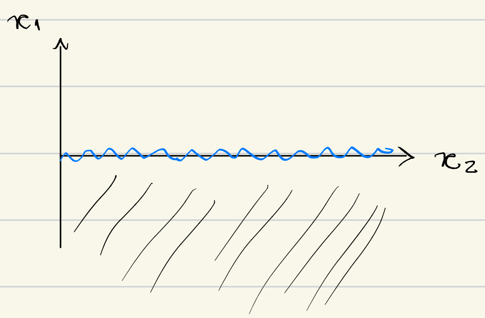
↳ gauge anomaly

↳ dynamical modes → cancel the anomaly coming from the bulk

$$M = \mathbb{R} \times \Sigma$$

$$\Sigma = (-\infty, 0) \times \mathbb{R}$$

$$\quad x_1 \qquad\quad x_2$$

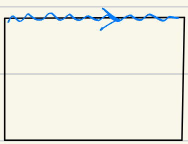


pure gauge conf.

$$a_\mu = \partial_\mu \phi$$

$$[\phi(x), \partial_y \phi(y)] = \frac{2\pi i}{K} \delta(x-y) \quad \text{(KM algebra)}$$

↳ commutator for a chiral boson (gapless)



$$\Pi(x) = \frac{K}{4\pi} \partial_x \phi$$

$$\mathcal{L} = \frac{K}{2\pi}\left(\partial_t \phi \, \partial_x \phi - v (\partial_x \phi)^2\right)$$

$$H = \int \frac{Kv}{4\pi} (\partial_x \phi)^2$$

$K > 0 \Rightarrow v > 0$ (left - propagation)

$K < 0 \Rightarrow v < 0$ (right - prop)

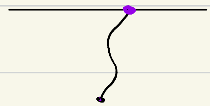
↳ Robust against perturbations

$$(\partial_t - v\,\partial_x)\,\partial x = 0$$

* Vertex operators

$$W_n = e^{in\int_{-\infty}^{0} dx'\, a_1}$$

$$= e^{in\phi}$$



$\phi \to 0$ when $x' \to -\infty$

$$\psi_n = :e^{in\phi}:$$

$$\psi_n \psi_m = e^{-i\pi/K\, nm\, \mathrm{sign}(\;)}\, \psi_m \psi_n$$

$\theta_n = \frac{n^2}{K}$ $\quad n = K$ odd $\Rightarrow \theta_{n=K}$ odd

fermionic

K- matrix formulation

$$S[a] = \int \frac{1}{4\pi} K_{IJ}\, \varepsilon^{\mu\nu\rho}\, a_\mu^I\, \partial_\nu\, a_\rho^J$$

N $U(1)$ emergent gauge fields a_μ^I, $I = 1, \dots, N$

K_{IJ} integer and symmetric matrix

$$\underbrace{K_{IJ}}_{\text{specifies CS coupling}} \in \mathbb{Z}^N \times \mathbb{Z}^N$$

→ TQFT

→ absence of local d.o.f

→ If K is diagonal → N decoupled theories

* Couplings

* EM field → induced current

filling fraction

$$S[a] = \int \frac{1}{4\pi} K_{IJ} \varepsilon^{\mu\nu\rho} a_\mu^I \partial_\nu a_\rho^J + A_\mu J^\mu$$

$$J^\mu = \frac{1}{2\pi} t_I \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho^I$$

t_I → vector charge $\in \mathbb{Z}^N$

$$\underline{J_{ind}^y} = \frac{1}{2\pi} \sigma^x \underline{E_x} \rightarrow \text{Hall effect}$$

$$\sigma = t_I (K^{-1})^{IJ} t_J \rightarrow \text{Hall conductance}$$

$$\rightarrow \boxed{\nu = t^T K^{-1} t} \, //$$

* Matter current

$$S[a] = \int \frac{1}{4\pi} K_{IJ} \varepsilon^{\mu\nu\rho} a_\mu^I \partial_\nu a_\rho^J - \frac{1}{2\pi} t_I \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho^I$$

$$- \underbrace{l_I}_{\text{qp vector } \in \mathbb{Z}^N} a_\mu^I j^\mu$$

$$j^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\rho} \partial_\nu \alpha_\rho$$

$$\mathcal{L}[A, c] = -t_I (K^{-1})^{IJ} l_J A_\mu j^\mu + \dots$$

$$\boxed{q_l = l^T K^{-1} l}$$

Charge - flux $\quad A_\mu = 0$

$$j^\mu{}_J = t_J j^\mu$$

$$j^\mu{}_J = \frac{1}{2\pi} K_{IJ} \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho^I$$

$b^I = 2\pi (K^{-1})^{IJ} \rho_J \to$ charge - flux

* Wilson lines $\to$ gauge invariant, non-local

$\hookrightarrow$ We get statistics $\hookrightarrow$ worldline of qp.

$$W_l[C] = \exp\left(i \int_C l_I a_i^I dx^i \right)$$

$$[a_I^i(x), a_J^j(y)] = 2\pi i (K^{-1})_{IJ} \varepsilon^{ij} \delta^2(\bar{x}-\bar{y})$$

$\hookrightarrow$ pair of conj fields

$$W_l[C] W_m[C'] = e^{i\theta} W_m[C'] W_l[C]$$

BCH theorem

$$\boxed{\theta_{l,m} = 2\pi l^T K^{-1} m}$$

$$\boxed{S_l = \pi \, l^T K^{-1} l}$$

↳ Local particles

* single - comp case ; $n = K$ (transp lines)
* In this case,

$$m_J = K_{IJ} \Lambda^I \rightarrow \text{physical observables}$$

$$\theta_{em} = 2\pi \underbrace{l_I \Lambda^I}_{\in \mathbb{Z}} = 2\pi \mathbb{Z}$$

$$q_m = t_I \Lambda^I \in \mathbb{Z}$$

$l \sim l + K\Lambda$ (superselection sectors)

↳ K matrix

* All diagonal elements are even → bosonic theory
* At least one odd diagonal element → fermionic

GSD → $|\det K|$ → # distinct qp's. → A

Ex

$$K = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \rightarrow \text{toric code}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \text{trivial sector}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \rightarrow \text{double semion}$$

→ Ex

$$K = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad t = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$r = t^T K^{-1} t = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

$|\det K| = 3 \qquad \text{rank } \mathcal{A} = 3$

qp's are defined mod local particles

$$l \sim l + K\Lambda \qquad ; \ \Lambda^T = (a, b)$$

$$l \sim l + \begin{bmatrix} 1 & \\ & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \sim l + \begin{pmatrix} a \\ -3b \end{pmatrix} \begin{matrix} \to \text{mod } 1 \\ \to \text{mod } 3 \end{matrix}$$

$$\mathcal{A} = \{(0,0), (0,1), (0,2)\} \qquad l = (0, l_2)$$

$$q_l = t^T K^{-1} l \Rightarrow q_l = \frac{-1}{3} l_2$$

$$\mathcal{A} = \{0, e/3, 2e/3\}$$

$$\theta_0 = 0$$

→ Edge theory

S_{bulk}

QM → dynamical modes

$$\mathcal{L}_{edge} = \frac{1}{4\pi} K_{IJ}\, \partial_t \phi^I \partial_x \phi^J - \frac{1}{2\pi} V_{IJ}\, \partial_x \phi^I \partial_x \phi^J$$

$V_{IJ} \rightarrow$ positive - definite

↳ signature of K (n_+, n_-)

$n_+ \rightarrow$ # positive eigenvalues $\rightarrow$ left - propag

$n_- \rightarrow$ # negative " $\rightarrow$ right - prop

$n_+ = n_- \rightarrow$ non - chiral

$n_+ \neq n_- \rightarrow$ net - chirality

* Wilson lines

$$\underbrace{\Psi_l = e^{i l^T \phi}} \qquad \underline{\text{gapless}}$$

* Perturbations

Incorporate the vertex operator into the Lagrangian

This can happen if Ψ_l is local

$$\Psi_l = e^{i l^T \phi} \rightarrow l^T \phi \text{ is } \boxed{\text{local}}$$

$$\underbrace{\Psi_m(x)}\, \Psi_l(y) = \exp\left(- [l^T \phi(x), l^T \phi(y)] \right) \Psi_l \Psi_m$$

$$[l^T \phi(x), l^T \phi(y)] = \pi i\, l^T K^{-1} l$$

$\nearrow$ $l^T K^{-1} l \in 2\mathbb{Z}$

↳ To localize the fields at classical values.

$\Rightarrow \boxed{l^T K^{-1} l = 0}$

if all modes are propagating to the same direction

= all eigenvalues of K are strictly positive

$\Rightarrow K^{-1}$ is positive definite

$$\forall\, l \neq 0 \Rightarrow \underline{l}^T \underbrace{K^{-1}} \underline{l} > 0$$

→ there are no perturbation that can gap the edge

↳ $n_+ = 2$ (right) $n_- = 1$

$$K = \left[\begin{array}{c|cc} \lambda_1 & & \\ \hline & -\lambda_2 & \\ & & \lambda_3 \end{array}\right]$$

Potential to be gapped

$$\rightarrow n_+ = n_- \text{ (non-chiral)}$$

→ Add perturbations

$$\frac{\tilde{C}_l}{2} e^{i l^T \phi} + \frac{\tilde{C}_l^*}{2} e^{-i l^T \phi} \sim \cos(l^T \phi + \alpha)$$

$\alpha = 0$ $\qquad m = K\Lambda$

$$\Psi_{K\Lambda}^{+} = e^{i \Lambda^T K \phi} \rightarrow \cos(\Lambda^T K \phi) \; ; \; l = K\Lambda$$

$\rightarrow \Lambda^T K \Lambda = 0$ Haldane

* To gap the edge

$\to n_+ = n_-$

$\Rightarrow \Lambda^T K \Lambda = 0$

$$K = \begin{bmatrix} 1 & \\ & -3 \end{bmatrix} \qquad \Lambda^T = (a, b)^T \qquad n_+ = n_-$$

$\Lambda^T K \Lambda = 0 \Rightarrow a^2 - 3b^2 = 0$

no integer solution

$\Rightarrow \cos(\Lambda^T K \phi) \to \Lambda^T_i K \phi = 2\pi n_i$

Lagrangian subgroup

$M \subset \mathcal{A}$

$M \Rightarrow$ Abelian group w/ fusion

1. $e^{i\theta_{m,m'}} = 1$, $m, m' \in M$

2. $l \in \mathcal{A}$, either

   $l = \sum_i c_i m_i \in M$

   or

   $l \notin M \to l$ has non-trivial with at least one $m \in M$

   $e^{i\theta_{lm}} \neq 1$, some m

$l \notin M \to$ confined

3. $m \times m' = m'' \in M, \quad m, m' \in M$

4. Bosonic systems

all particles in M are bosonic

$$e^{i\theta_m} = 1, \ \forall\, m \in M$$

↳ $\Lambda^T_i K \Lambda_j = 0$ iff exists a $\mathcal{L}S$.

↳ Are capable of classifying distinct gapped edge theories

$$K = \begin{bmatrix} 1 & 0 \\ 0 & -9 \end{bmatrix} \qquad t = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{fermionic}$$

$$\nu = 1 - \frac{1}{9} = \frac{8}{9} \qquad n_+ = n_-$$

$$|\det K| = 9 \qquad l \sim l + \begin{bmatrix} n \\ 9m \end{bmatrix}$$

$$\mathcal{A} = \{(1,0), (1,1), \ldots, (1,8)\}$$

$$S(1,0) = \pi = S(1,6)$$

$$S(1,3) = 0$$

$$\Lambda^T K \Lambda = 0 \qquad \Lambda^T = (a,b)$$

$$a^2 - 9b^2 = 0$$

$$\Lambda = (3, \pm 1)$$

→ Lagrangian sub.

$$M = \{(1,0),\ (1,3),\ (1,6)\} \qquad V = \begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix}$$

$\hookrightarrow$ K matrix

$$\mathcal{L}_1 = \frac{1}{4\pi}\,\partial_t\phi^1\,\partial_x\phi^1 - \frac{v_1}{4\pi}\,(\partial_x\phi^1)^2$$

$$\mathcal{L}_2 = -\frac{9}{4\pi}\,\partial_t\phi^2\,\partial_x\phi^2 - \frac{v_2}{4\pi}\,(\partial_x\phi^2)^2$$

$$\psi_I^\dagger = e^{i\underbrace{K_{IJ}\phi^J}} \qquad \phi = (\phi^1, \phi^2)$$

$$\psi_1^\dagger = \underline{e^{i\phi^1}}\,, \qquad \psi_2^\dagger = e^{-9i\phi^2}$$

$$\cos(\Lambda^T K \phi)\,, \qquad \Lambda^T = (3, \pm 1)$$

$$\cos[3(\phi^1 \pm 3\phi^2)]$$

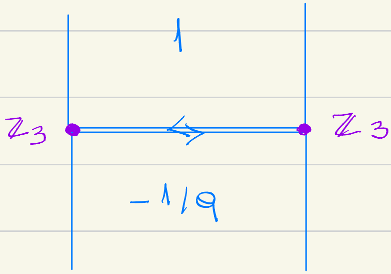
$$SC \to \Lambda^T = (3,1)$$

$$\cos[3(\phi^1 - 3\phi^2)] \sim e^{i(3\phi^1 - 9\phi^2)} + h.c$$

$e^{3i\phi^1} = (\psi_1^\dagger)^3 \leftarrow$ 3 electrons in mode 1

$e^{-9i\phi^2} = (\psi_2)^\dagger \leftarrow$ 1 electron in mode 2

$$\Lambda^T K \phi = c_i\, m^T \Rightarrow m^T = (1,-3) = (1,6)$$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Z3
Z3
-1/9