Sandwich construction of (2+1)d Toric code

References

- arxiv 2511.19793 → Xie chen
- arxiv 2404.19004 → fermion condensation

The presentation

1) General idea
   - sketch the mechanism (symTFT)
   - Toric code
   - Transverse field Ising model (TFIM)
   - Superconductor

2) Bosonic boundaries
   - Top boundary
   - bottom boundary
   - Phases of TFIM

3) Fermionic boundaries
   - Jordan-Wigner
   - fermionic condensation
   - bottom boundaries
   - General picture

1) General idea

Sym TFT setup

(reference)
symmetry boundary

(D+1)d Topological order

physical boundary
(dynamical)

condensing on the top boundary

: symmetry operators

charged objects

condensing on the bottom boundary

: Hamiltonian

$\mathbb{Z}_2$ gauge : $a = \{c, \mathrm{Rep}\,\mathbb{Z}_2\}$

$\mathbb{Z}_2 : \{1, g\}$

$1 = \{1, 1\}$

$e = \{1, g\}$ } each one is a phase

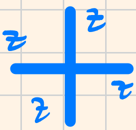
$m = \{g, 1\}$

$f = \{g, g\}$

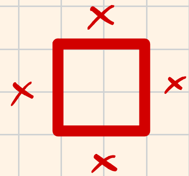
gauge $\mathbb{Z}_2 \longrightarrow \text{Rep}\, \mathbb{Z}_2$

- Toric code

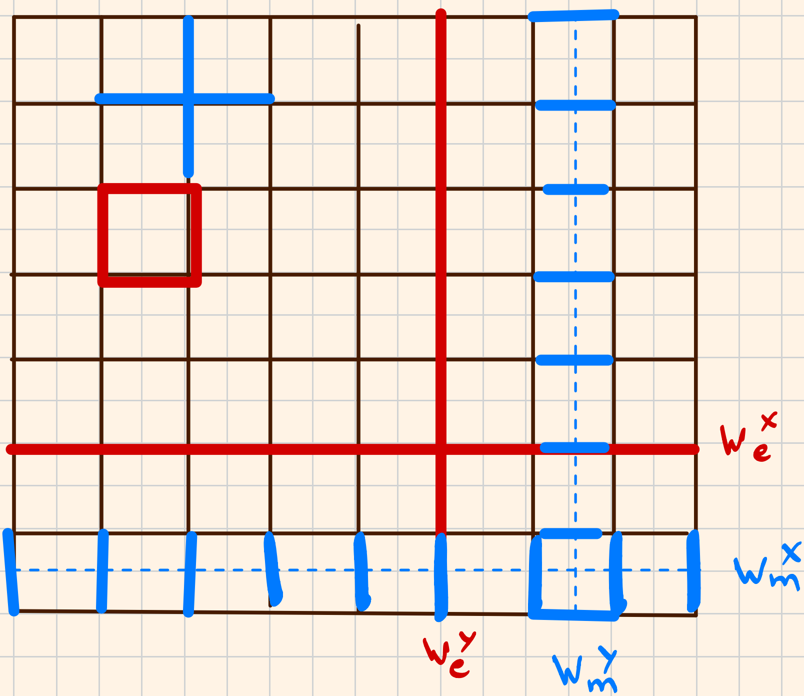
$$H_{TC} = -\sum_s A_s - h \sum_p B_p$$



$$A_s = \prod_{e \ni s} Z_e \,, \qquad B_p = \prod_{e \in p} X_e$$



$\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$ symmetry: $W_e = \prod_{e \in \gamma} X_e \,, \quad W_m = \prod_{e \in \tilde{\gamma}} Z_e$



$$G \longmapsto \hat{G} = \text{Rep}(G)$$

if G is abelian $\quad \text{Rep}\, G = G$

- TFIM

$$H_{\text{Ising}} = -(1-\lambda)\sum_i \sigma_i^z \sigma_{i+1}^z - \lambda \sum_i \sigma_i^x$$

$\mathbb{Z}_2^{(0)}$ symmetry : $U = \prod_i \sigma_i^x$

Phases

$\lambda = 1$: $\mathbb{Z}_2$ symmetric (para)

$\lambda = 0$: $\mathbb{Z}_2$ SSB (ferro)

$\lambda = \frac{1}{2}$: Ising CFT or critical

- Superconductor

$$H_{\text{sup}} = -(1-\lambda)\sum_j i\gamma_{2j}\gamma_{2j+1} - \lambda \sum_j i\gamma_{2j-1}\gamma_{2j}$$

Majorana fermions

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \gamma_i = \gamma_i^\dagger, \quad \gamma_i^2 = 1$$

fermion parity : $P_f = \prod_j i\gamma_{2j-1}\gamma_{2j}$

charged op : $c = \dfrac{\gamma_{2j-1} + i\gamma_{2j}}{2}$

complex fermion

Phases:

$\lambda = 1$: Trivial superconductor

$\lambda = 0$: topological superconductor

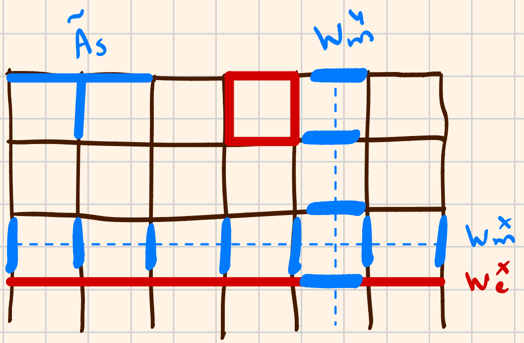
$\lambda = \frac{1}{2}$: massless majorana

2) Bosonic boundaries

TC: $\{1, e, m, f = em\}$

• Top boundaries

• smooth: W_m condenses



• stabilizer $\tilde{A}_s = 1$

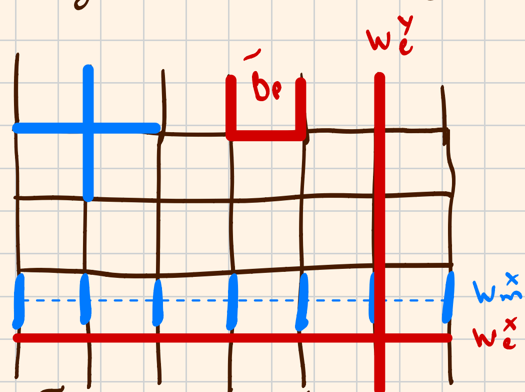
$\prod \tilde{A}_s = W_m^x = 1$

• W_e^x is free $\rightarrow$ symmetry op.

• W_m^y is charged op.

• W_e^y

• Rough : W_e condenses



• $\tilde{B}_p = 1 \Rightarrow \prod \tilde{B}_p = W_e^x = 1$

• $W_m^x \rightarrow$ sym. op

• charged op: W_e^y

$\mathbb{Z}_2$ sym: $W_e^x = \prod X \to U = \prod \sigma^x$

charged op: $W_m^y = \prod Z \to \sigma^z$

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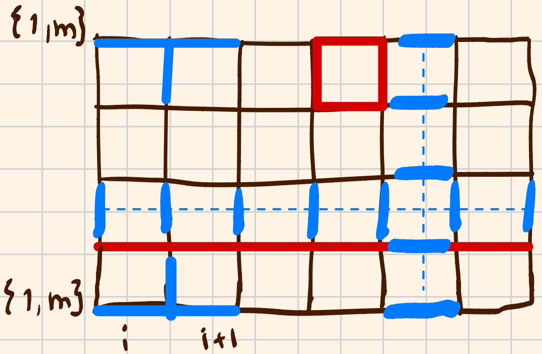
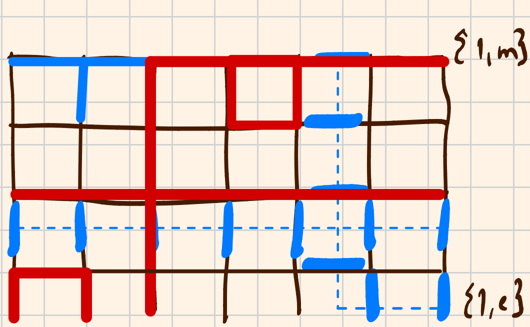
charged op: $W_e^y = \prod X \to \sigma^x$

• Bottom boundaries





Smooth bottom:

stabilizer: $\tilde{A}_s \longrightarrow \sigma_i^z \sigma_{i+1}^z$

local op: $W_m^y \longrightarrow \sigma_i^z$

$$\boxed{H_{SSB} = -\sum_i \sigma_i^z \sigma_{i+1}^z}$$

Rough bottom:

stabilizer: $\tilde{B}_p \longrightarrow \sigma_i^x \sigma_{i+1}^x$

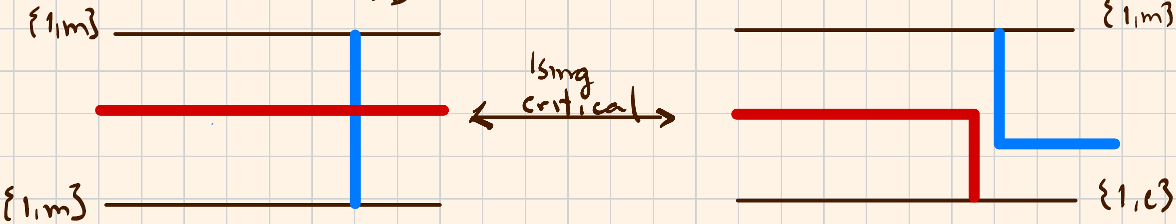
local operator: $W_e^y \to \sigma^x$

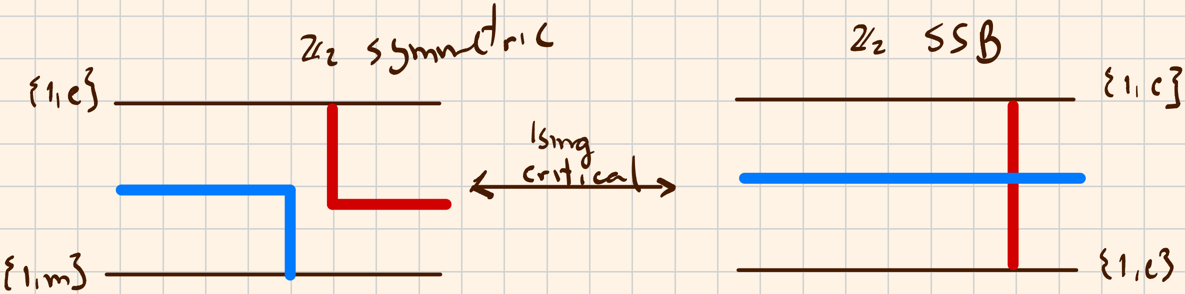
$$\boxed{H_{para} = -\sum_i \sigma_i^x}$$

• Phases of TFIM

$\mathbb{Z}_2$ SSB

$\mathbb{Z}_2$ symmetric





changing the top boundary is equivalent to KW