Sandwich construction of (2+1)d Toric code

References

- arxiv 2511.19793 → Xie chen
- arxiv 2404.19004 → fermion condensation

The presentation

1) General idea
   - sketch the mechanism (symTFT)
   - Toric code
   - Transverse field Ising model (TFIM)
   - Superconductor

2) Bosonic boundaries
   - Top boundary
   - bottom boundary
   - Phases of TFIM

3) Fermionic boundaries
   - Jordan - Wigner
   - fermionic condensation
   - bottom boundaries
   - General picture

1) General Idea

Sym TFT setup

(reference)
symmetry boundary

(D+1)d Topological order

physical boundary
(dynamical)

condensing on the top boundary

● : symmetry operators
charged objects

condensing on the bottom boundary

● : Hamiltonian

$\mathbb{Z}_2$ gauge : $a = \{c, \mathrm{Rep}\,\mathbb{Z}_2\}$

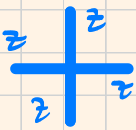
$\mathbb{Z}_2 : \{1, g\}$

$$\left.\begin{aligned} 1 &= \{1, 1\} \\ e &= \{1, g\} \\ m &= \{g, 1\} \\ f &= \{g, g\} \end{aligned}\right\} \text{each one is a phase}$$

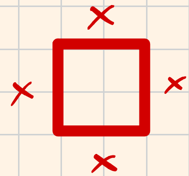
• Toric code

gauge $\mathbb{Z}_2 \longrightarrow \text{Rep}\, \mathbb{Z}_2$

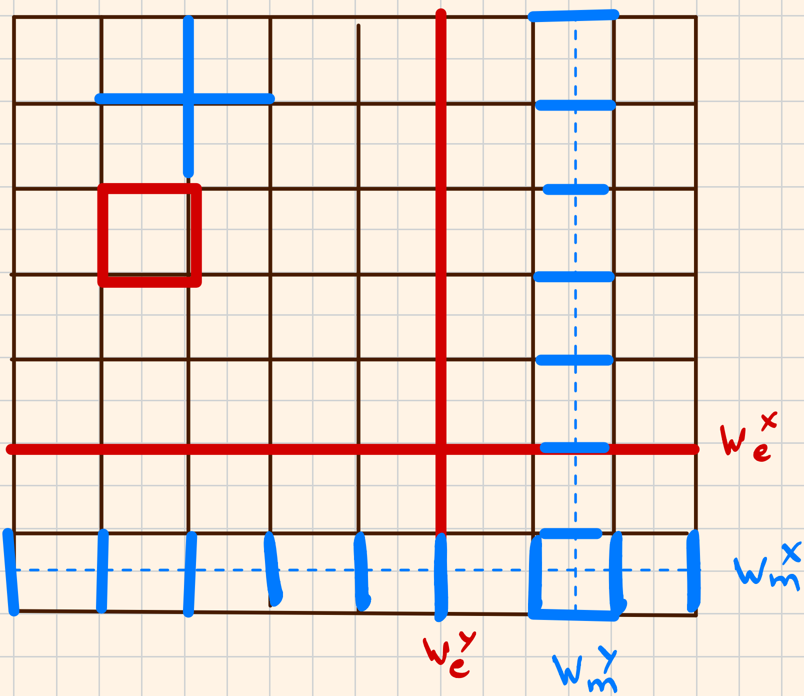
$$H_{TC} = -\sum_s A_s - h \sum_p B_p$$



$$A_s = \prod_{e \ni s} Z_e \, , \qquad B_p = \prod_{e \in p} X_e$$



$$\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)} \text{ symmetry: } W_e = \prod_{e \in \gamma} X_e \, , \quad W_m = \prod_{e \in \tilde{\gamma}} Z_e$$



$$G \longmapsto \hat{G} = \text{Rep}(G)$$

if G is abelian $\quad \text{Rep}\, G = G$

- TFIM

$$H_{\text{Ising}} = -(1-\lambda)\sum_i \sigma_i^z \sigma_{i+1}^z - \lambda \sum_i \sigma_i^x$$

$\mathbb{Z}_2^{(0)}$ symmetry : $U = \prod_i \sigma_i^x$

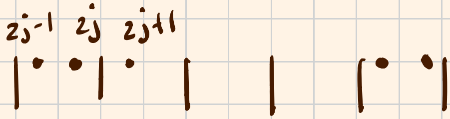
Phases

$\lambda = 1$: $\mathbb{Z}_2$ symmetric (para)

$\lambda = 0$: $\mathbb{Z}_2$ SSB (ferro)

$\lambda = \frac{1}{2}$: Ising CFT or critical

- Superconductor



Majorana fermions

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \gamma_i = \gamma_i^\dagger, \quad \gamma_i^2 = 1$$

fermion parity : $P_f = \prod_j i\,\gamma_{2j-1}\gamma_{2j}$

charge du op : $c = \dfrac{\gamma_{2j-1} + i\gamma_{2j}}{2}$

complex fermion

Phases:

$\lambda = 1$: Trivial superconductor

$\lambda = 0$: topological superconductor

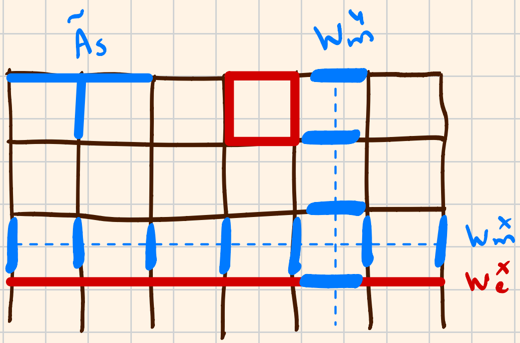
$\lambda = \frac{1}{2}$: massless majorana

2) Bosonic boundaries

TC: $\{1, e, m, f = em\}$

• Top boundaries

• Smooth: W_m condenses



• stabilizer $\tilde{A}_s = 1$

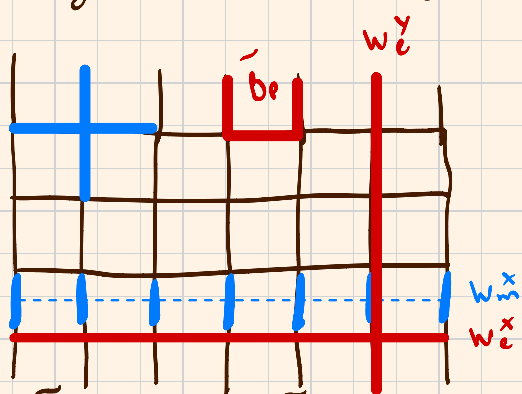
$\prod \tilde{A}_s = W_m^x = 1$

• W_e^x is free → symmetry op.

• W_m^y is charged op.

• W_e^y

• Rough: W_e condenses



• $\tilde{B}_p = 1 \;=\; \prod \tilde{B}_p = W_e^x = 1$

• W_m^x → sym. op

• charged op: W_e^y

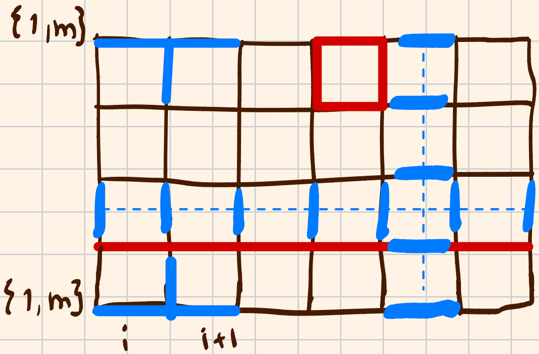
$\mathbb{Z}_2$ sym: $W_e^x = \prod X \to U = \prod \sigma^x$

charged op: $W_m^y = \prod Z \to \sigma^z$

$\mathbb{Z}_2$ sym: $W_m^x = \prod Z \to U = \prod \sigma^z$

charged op: $W_e^y = \prod X \to \sigma^x$

• Bottom boundaries

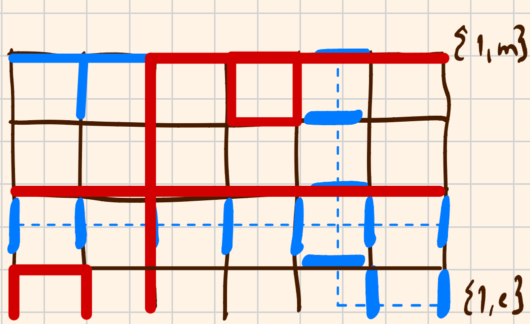


Smooth bottom:

stabilizer: $\tilde{A}_s \longrightarrow \sigma_i^z \sigma_{i+1}^z$

local op: $W_m^y \longrightarrow \sigma_i^z$

$$\boxed{H_{SSB} = -\sum_i \sigma_i^z \sigma_{i+1}^z}$$



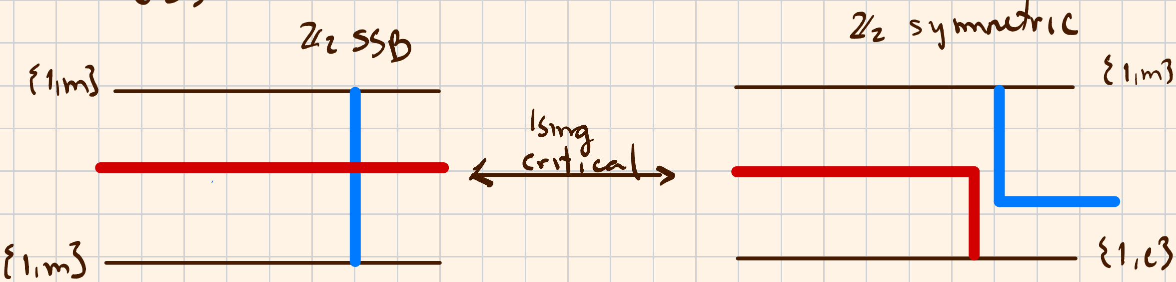
Rough bottom:

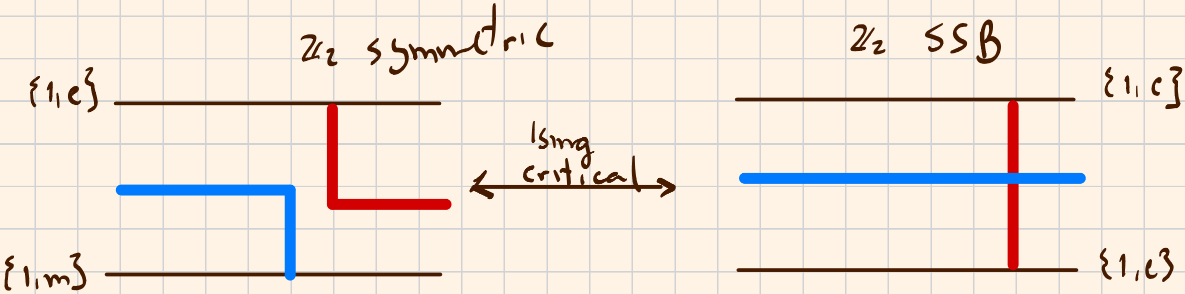
stabilizer: $\tilde{B}_p \longrightarrow \sigma_i^x \sigma_{i+1}^x$

local operator: $W_e^y \to \sigma^x$

$$\boxed{H_{para} = -\sum_i \sigma_i^x}$$

• Phases of TFIM





changing the top boundary is equivalent to KW

$\mathbb{Z}_2$ gauge theory in $(1+1)d$ $\sim$ TFIM

$$H = -\sum_i Z_i Z_{i+1} - g \sum_i X_i$$

$\mathbb{Z}_2$ gauge theory in $(2+1)d$ $\sim$ Toric code
↳ deconfined

$$H = -\sum_p B_p - h \sum_s A_s$$

$(1, g) \to (1, \mathrm{Rep}(g))$

Lagrangian algebras : $\mathcal{A}_e = \{1, e\}$

$\mathcal{A}_m = \{1, m\}$

$(g, \mathrm{Rep}(g))$

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3) Fermionic boundaries

- Jordan - Wigner transf

$$H_{sup} = -(1-\lambda)\sum_j i\,\gamma_{2j}\,\gamma_{2j+1} - \lambda \sum_j i\,\gamma_{2j-1}\,\gamma_{2j}$$

$$\Big\updownarrow \text{ JW transf.}$$

$$H_{Ising} = -(1-\lambda)\sum_i \sigma_i^z \sigma_{i+1}^z - \lambda \sum_i \sigma_i^x$$

JW:

$$\gamma_{2j-1} = \sigma_j^z \prod_{k<j} \sigma_k^x$$

$$\gamma_{2j} = \sigma_j^y \prod_{k<j} \sigma_k^x$$

$$\gamma_{2j+1} = \sigma_{j+1}^z \prod_{k<j+1} \sigma_k^x$$

$$\sigma^y \sigma^x = i\varepsilon^{yxz}\sigma^z$$

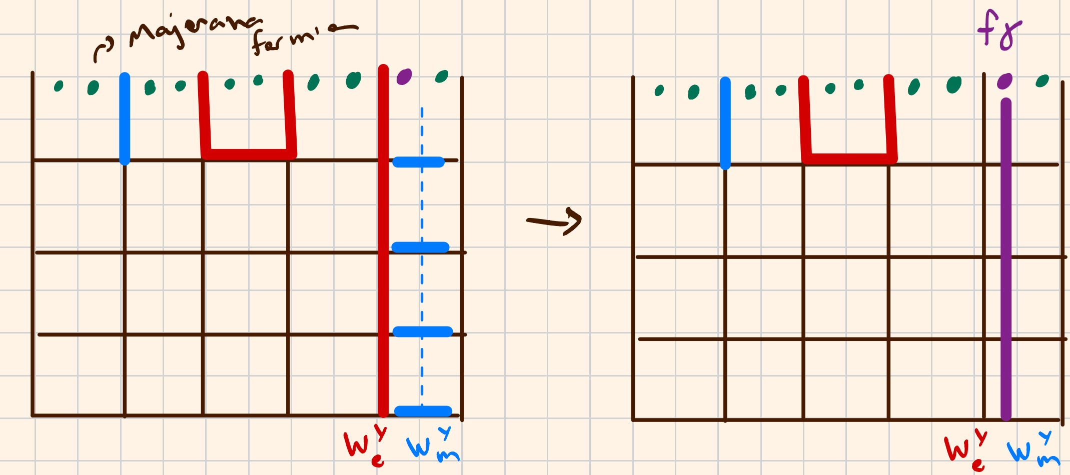
$$\gamma_{2j}\,\gamma_{2j+1} = \sigma_j^y \prod_{k<j} \sigma_k^x \;\; \sigma_{j+1}^z \prod_{k<j+1} \sigma_k^x$$

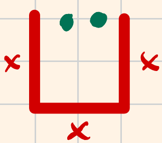
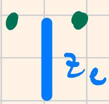
$$= \sigma_j^y\,\sigma_{j+1}^z\,\sigma_j^x = -i\,\sigma_{j+1}^z\,\sigma_j^z$$

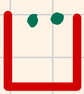
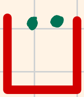
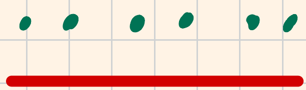
$$P_f = \prod_j i\gamma_{2j-1}\gamma_{2j} \longrightarrow U = \prod_j \sigma^x_j$$

fermion parity sym. $\qquad\qquad$ $\mathbb{Z}_2$ sym.

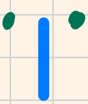
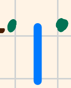
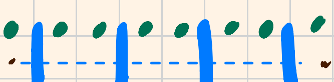
• Fermion condensation



 $= i\gamma_{2j-1}\gamma_{2j}\tilde{B}_p$ $\qquad$  $= i\gamma_{2j}\gamma_{2j+1} Z_e$

 $= 1 \Rightarrow \prod$  $= \cdots$  $\cdots = W_e^x P_f = 1$

$W_e^x = \prod_i \sigma_i^x, \quad P_f = \prod_j i\gamma_{2j-1}\gamma_{2j}$

 $= 1 \Rightarrow \prod$  $= \cdots$  $\cdots = W_m^x \tilde{P}_f$

$W_m^x = \prod_i \sigma_i^z, \quad \tilde{P}_f = \prod_i i\gamma_{2j}\gamma_{2j+1}$

from stabilizer condition

$W_e^x = P_f$

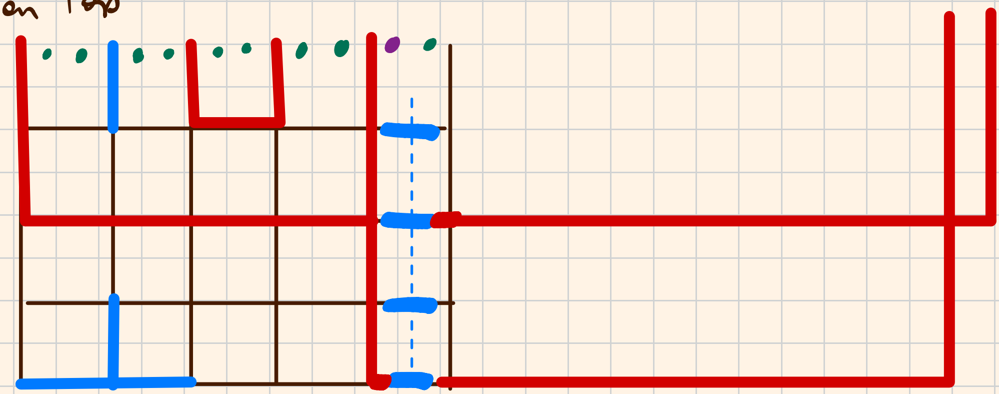
↓ bulk ↳ boundary

symmetry op: W_e^x

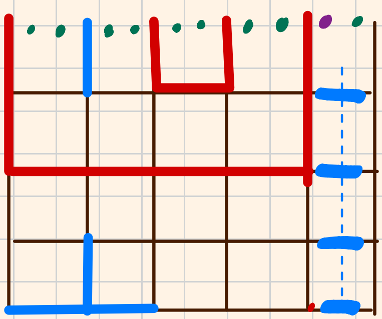
charged obj: W_f

- bottom boundary

fermion top



smooth bottom



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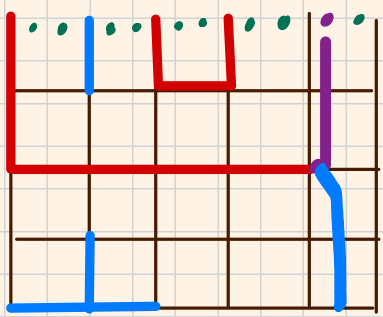
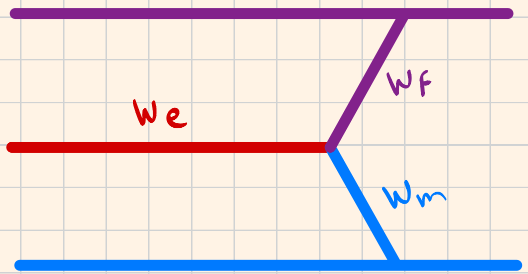




Topological superconductor

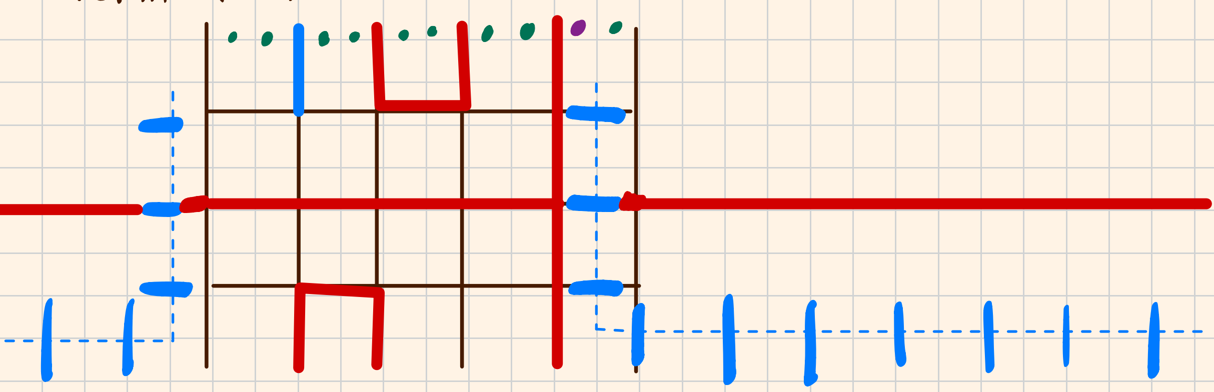
smooth bottom Hamiltonian

stabilizer: $\tilde{A}_s = \underset{z_{2j} \quad z_{2j+1}}{\perp^{z_e}} = 1 \Rightarrow z_e = z_{2j} z_{2j+1}$

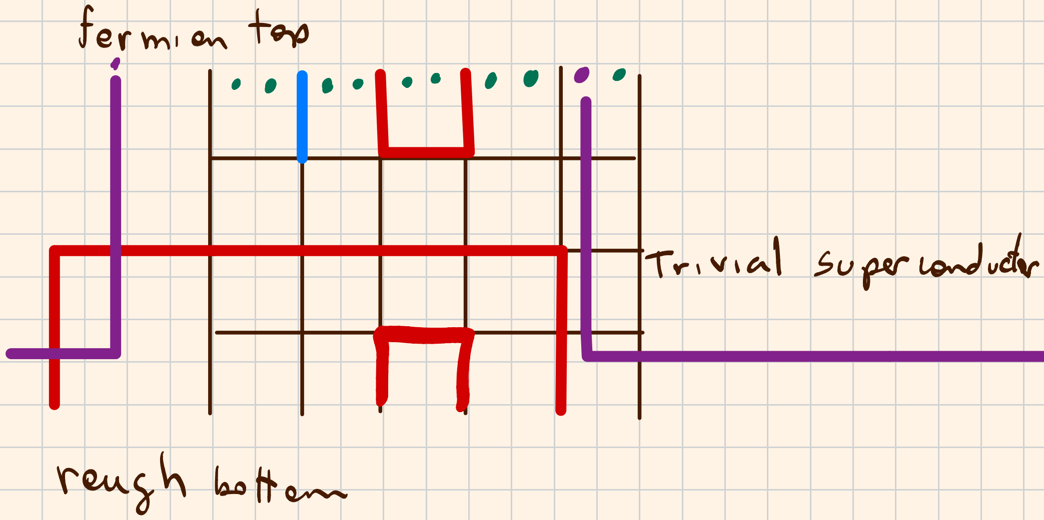
$$i \gamma_{2j} \gamma_{2j+1} z_e = 1 \Rightarrow z_e = i \gamma_{2j} \gamma_{2j+1}$$

$$\boxed{H_{TC} = -\sum_j i \gamma_{2j} \gamma_{2j+1}}$$

fermion top



Rough bottom



stabilizer: $\tilde{B}_p = 1 \rightarrow \tilde{B}_p \, i\gamma_{2j-1}\gamma_{2j} = 1$

$$\tilde{B}_p = i\gamma_{2j-1}\gamma_{2j}$$

$$\boxed{H_{\text{Trivial}} = -\sum_j i\gamma_{2j-1}\gamma_{2j}}$$