Lecture #2

i) Materials @ gui-delfino.github.io

ii) Homework

$$H = -\frac{\Delta_e}{2}\sum_s A_s - \frac{\Delta_m}{2}\sum_p B_p$$

Δ_e ~ electric mass

$\Delta_e, \Delta_m \to \infty$

Δ_m ~ magnetic mass

In the continuum: $S = \frac{2}{2\pi}\int a \wedge db \to H = 0$

describes low-energy physics of toric code

Recap: Let G a finite group. Representations are homomorphism

$$\rho : G \longrightarrow GL(n, \mathbb{C})$$

dimension.

Irreps = "Irreducible representations".

("simple objects" $\neq \oplus$ objects

i) there are r Irreps

$r = \#$ conjugacy classes

Conjugacy class is a equivalence class which is defined by the equivalence relation:

$$h \sim g \quad \text{if } \exists K \in G \text{ s.t. } g = K^{-1} h K$$

ii) If $n_1, n_2, \dots, n_r$ the dimensions of the Irreps of G

$$\sum_{i=1}^{r} n_i^2 = |G|$$

$1, 1, 2$

$$\rightsquigarrow 1 + 1 + 4 = 6 = |S_3|$$

<u>Maskke's theorem</u>: Every representation of a finite group is a direct sum of Irreps.

Def: Let
$g: G \to GL(V)$ be representation
$\phi: G \to GL(W)$

A intertwiner T maps one representation to the another one
$T\rho(g) = \phi(g)T \quad \forall g \in G$

If T is invertible (linear) $\phi(g) = T\rho(g)T^{-1} \quad \forall g \in G$
then g is said to be equivalent to ϕ

Forms → Representation → Categories

Def: A category $\mathcal{C}$ is a collection of objects $ob(\mathcal{C})$ (or $Ob_{\mathcal{C}}(\mathcal{C})$)
together with collection of morphisms $Hom(X,Y)$ for any two objects $X, Y \in ob(\mathcal{C})$ (hom space)

such that:
i) $\forall X \in ob(\mathcal{C})$, $\exists\, id_X \in Hom(X,X)$ → identity morphism

ii) Composition $f \in Hom(X,Y)$ $f: X \to Y$
and $g \in Hom(Y,Z)$ $g: Y \to Z$
$g \circ f \in Hom(X,Z)$

String diagram

$f: X \to Y$
or
$f \in Hom(X,Y)$

i) Identity morphism

ii) Composition

Example:
→ $\mathcal{C} = \mathbf{Set}$:
- objects in $\mathcal{C}$ are finite sets
- morphisms are functions

→ $\mathcal{C} = \mathbf{Grp}$
- objects in $\mathcal{C}$ are finite groups
- morphisms are group homomorphisms

⋮

$\mathbf{Top}$, $\mathbf{Man}$, ...

So many "kind" categories (and depend on $\mathcal{C}$ is still an objective structure).

More interesting for us, we work "triple" categories.

Ex: Category associated to a finite group G
("category" G)
- $\mathcal{C} = Cat(G)$ where $ob_{\mathcal{C}}(\mathcal{C})$ are group elements
- $Hom(g,h) = id_g$ if $g = h$, empty otherwise

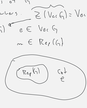
If get →
and if get $Hom(g,h) \neq \emptyset$

→ $Vec(G)$ is another way to categorify a finite group G

→ Now we use the "category" representations of G
$\mathcal{C} = \mathbf{Rep}(G)$ where
$ob_{\mathcal{C}}(\mathcal{C})$ are representations of G Dimensional vector
and morphisms are intertwiners $\underbrace{\mathcal{C}(Vec_G) = Vec_G \otimes Rep(G)}$
Kitaev's Quantum Double $D(G) = \mathcal{C} \in Vec_G$
$\cong Rep_f(G)$

$a, b, c, \ldots \in ob(\mathcal{C})$
We want to understand how to make sense of both
- fusing anyons
- multiplying anyons

(by fusion) $\boxed{a \times b = \sum_c N_c^{ab}\, c}$



Fusion categories = "categorify fusion rings"

Fusion rings

set of elements R

Def: A unital ring $(R,+,\cdot)$ such that
i) $(R,+)$ is a Abelian group with identity 0
ii) multiplication $\cdot$ is associative
$r_1 \cdot (r_2 \cdot r_3) = (r_1 \cdot r_2) \cdot r_3 \quad \forall r_1, r_2, r_3 \in R$
iii) Distributivity
$r \cdot (s+t) = r \cdot s + r \cdot t \quad \forall r, s, t \in R$
iv) $\exists$ multiplicative identity $\underline{1}$ st. $r \cdot 1 = r \quad \forall r \in R$
(unital)

Ex: Matrices $n \times n$ form a ring
↳ makes sense of right and left multiplication
However, not necessarily invertible.

Group: $(G, \square)$
i) $\exists\, 1 \in G$ st $g \square 1 = g$
ii) $\exists$ (unique) g^{-1} st $g^{-1} \square g = 1$
iii) Associativity

Example: • $(\{\text{Matrices } n \times n\}, +)$ is an Abelian group (Inv.)
Neutral element $0_{n \times n}$
Inverse of M is $-M$

• $(\{\text{Inv. Matrices } n \times n\}, \cdot)$ is non-Abelian
Neutral element $\mathbb{1}_{n \times n}$
Inverse of M is M^{-1}

⇒ ring = $(\{\text{Matrices } M_{n \times n}\}, +, \cdot)$

Def: A fusion ring is a unital ring $(F, +, \times)$ which
i) is a free $\mathbb{Z}$-module with a finite basis $L \subset F$
($\exists$ LCF such that every element in F can be written as a finite integer linear combination of elements in L)
$y = n_1 a_1 + n_2 a_2 + \ldots$ for $a_1, a_2, \ldots \in L$

e.g.: $Rep(S_3) \Rightarrow L = \{triv, sgn, std\}$
↳ elements of L are Irreps.

ii) there is a involution $*: L \to L$ (called "duality")
such that
i) $\mathbb{1} \in L$, $\mathbb{1}^* = \mathbb{1}$
ii) $a \times b = \sum_{c \in L} N_c^{ab}\, c$, $N_c^{ab} \in \mathbb{Z}_{\geq 0}$ $\forall a, b, c \in L$
iii) $N_{\mathbb{1}}^{ab} = N_{\mathbb{1}}^{ba} = \delta_{a^* b}$
a^* is the "anti-particle" of a
$a \times a^* = \mathbb{1} + \ldots$
(can be extended to elements in F)
$(a \times b)^* = b^* \times a^*$

Remark:
- Rank of a fusion ring $|L|$
- Notation $(L, \{N_c^{ab}\})$
- In general $a \times b \neq b \times a$

For fusion rings st $a \times b = b \times a$
(commutative fusion ring (used to describe anyons))

$A \times B = B \times A = 1 + C + D + \ldots$

Ex: $L = \{1, g, g^2\}$ or $|L| = 3$
$g^2 \times g = 1$ $\quad g^2$
$g^* = g$ $\quad g \times g^2 \times g = g$

$L = \{1, \tau\}$
Ex: $\tau \times \tau = 1 + \tau$
$\tau^* = \tau$ $\quad$ or $|L| = 2$

$Y = \tau \times \tau \times \tau = n_1 1 + n_2 \tau$
2τ

$\tau \times \tau \times \tau = 1 + 5\tau$

$|Hom(\tau \times \tau \times \tau, \tau)| = 5$