$X \in \mathcal{F}$

$\alpha_X, \iota_X \in \mathbb{C}$, $\alpha_X \iota_X = 1$

$X \to \alpha_X \iota_X X$

$\Rightarrow \Sigma = \mathbb{R}$

$(\iota, \iota \alpha^2 \iota) \xrightarrow{\text{Frob}} \text{Frob. subspace} \xrightarrow{\text{Perron}} (\iota, \iota \alpha^2 \iota)$ unique

Subring 3

Fusion Ring $(\mathbb{L}, \iota \alpha^2 \iota)$
$\mathbb{L}$ skel set
closed

→ any closed $X \in \mathcal{F} \sim$ to $\mathbb{L}$ shaped $\alpha \iota$ Frob

$X = \sum_{i \in \mathbb{L}} n_i x_i$ for $n_i \in \mathbb{Z}_{\geq 0}$

→ Z-subring ring $\alpha \iota \alpha \iota$ (stably real)

$\exists\ 1 \in \mathbb{L}$, $1 \alpha 1 = 1$

i) $\iota \alpha \iota = \sum_{i \in \mathbb{L}} N_{ij}^k \iota_k$, $N_{ij}^k \in \mathbb{Z}_{\geq 0}$

ii) $N_{ij}^k = N_{ji}^k$ - Sym as $\iota \alpha \iota = \iota \alpha \iota$

$\Rightarrow$ Multiplicative id $N_{1j}^k \in \{0,1\}$
$\mathbb{L} = \{1, \iota, \ldots\}$ where $\iota \alpha \iota = 1$

$1 \alpha 1 = 1$
$1 \alpha \iota = \iota$
$\iota \alpha 1 = \iota$
$\iota \alpha \iota = 1 + \ldots$

$\Rightarrow$ Fusion Ring (FR) has unique $\iota \in \mathbb{L}$ such $\iota \alpha \iota$
(Longest positive eigenvalue of fusion matrix $(N_{ij})_{kl} = N_{ij}^k$)

$\Rightarrow$ Note: $\sum_{k \in \mathbb{L}} N_{ij}^k = 1$

$\iota \alpha \iota = \sum_{k \in \mathbb{L}} N_{ij}^k \iota_k$

Ex $\mathbb{L} = \{1, \iota, \iota \alpha \iota = 1 + \iota\}$

i) $1 \alpha 1 = 1$
ii) $1 \alpha \iota = \iota \alpha 1 = \iota$ (left-right unit law)
iii) $\iota \alpha \iota = 1 + \iota$ (left-right unit law)

$\Rightarrow \begin{cases} 1 \alpha 1 = 1 \\ 1 \alpha \iota = \iota \\ \iota \alpha 1 = \iota \\ \iota \alpha \iota = 1 + \iota \end{cases}$

$\Rightarrow \begin{cases} d_1 \alpha d_1 = 1 \\ d_1 \alpha d_\iota = d_\iota \\ d_\iota \alpha d_1 = d_\iota \\ d_\iota \alpha d_\iota = 1 + d_\iota \end{cases}$

$\Rightarrow$ If $d_\iota \neq 1$ then ι is non-A obs

FR because d_ι are finite terms

If $d_\iota \neq 1$ $\Rightarrow$ $\exists$ fractional degrees of freedom $\Rightarrow$ $\mathcal{F}$ is sub-theoretical unstable (anyons)

→ displayed finite complexity $\Rightarrow$ sub-Fibonacci anyons

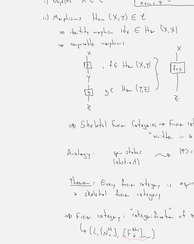
Universal unitary category $\exists$ $\mathcal{F}$ s.t. $\mathrm{Hom}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$
i.e. - Ising anyons $\mathbb{L} = \{1, \psi, \sigma\}$ $\mathcal{F}$-Ising-like

$\left.\begin{array}{l} \psi \alpha \psi = 1 \\ \psi \alpha \sigma = \sigma \\ \sigma \alpha \sigma = 1 \end{array}\right\}$ non-universal

- Fibonacci anyons → Universal

Constants:
unital: $N_{1j}^k = \delta_{kj}$, $N_{j1}^k = \delta_{kj}$
involutive: $N_{ij}^1 = \delta_{ij} \alpha^*$, $N_{ij}^1 = N_{ji}^1 \alpha N_{ij}^1$

Associativity: $\sum_{s \in \mathbb{L}} N_{ij}^s N_{sk}^l = \sum_{t \in \mathbb{L}} N_{jk}^t N_{it}^l$
(Verlinde's formula)



$\Rightarrow N_{ij}^k$ are $|\mathbb{L}|$ elements
which are not all independent

$\Rightarrow$ Classify all fusion rings of rank $|\mathbb{L}|$
by solving the constraints (Algebra Watch)

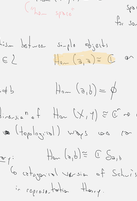
$\Rightarrow$ Multiplicity free $\Leftrightarrow$ to rank $\mathbb{L}$ Grothendieck

Fusion Categories

Category $\mathcal{C}$

i) Objects $X \in \mathcal{C}$

ii) Morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathcal{C}$
$\Rightarrow$ identity morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$
$\Rightarrow$ composable morphisms



$\Rightarrow$ Skeletal Fusion Category $\Rightarrow$ Fusion category
- unitor = α basis

Analogy: spin states $\rightsquigarrow$ $|\Psi\rangle = \begin{pmatrix} \iota \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \iota \end{pmatrix}$
(vectors)

Theorem: Every basic category is equivalent to a skeletal basic category

$\Rightarrow$ Fusion category: categorification of a fusion ring
(e.g. $(\mathbb{L}, \iota \alpha^2 \iota)$ $\{\mathbb{1}, \psi\}$ _) $\qquad$ ($\mathbb{L}, \iota \alpha^2 \iota$)

$\Rightarrow$ Now ingredients: Fusion

Let us create a fusion ring $(\mathbb{L}, \iota \alpha^2 \iota)$, then use a construct a fusion category $\mathcal{C}$ out of it

i) Objects
Any object in $\mathcal{C}$ is of the form
$X = \bigoplus_{a \in \mathbb{L}} n_a \, a$, $n_a \in \mathbb{Z}_{\geq 0}$

$\Rightarrow$ the elements of $\mathbb{L}$ are called "simple objects"

$\Rightarrow$ the rank of $\mathcal{C}$ is $|\mathbb{L}|$

Fusion category allows us to multiply (or fuse) objects

$a \otimes b = \sum_{c \in \mathbb{L}} N_{ab}^c \, c$

Notation: - numerals letters $1, \alpha, \iota, \ldots$ - simple objects
- Uppercase letters $X, Y, \ldots$ to mean general objects

Morphisms
Consider - morphisms between $X, Y \in \mathcal{C}$

$\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathbb{C}^n$ ~ n-dimensional vector spaces over $\mathbb{C}$
(Hom space)
the same n

i) Morphisms between simple objects
$a \in \mathbb{L}$ $\qquad \mathrm{Hom}_{\mathcal{C}}(a, a) \cong \mathbb{C}$ or $\mathrm{span}_{\mathbb{C}}(\mathrm{id}_a)$

If $a \neq b$ $\qquad \mathrm{Hom}_{\mathcal{C}}(a, b) \cong \phi$

The dimension of $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathbb{C}$ counts how many (topologically) ways can go X to Y

Summary: $\mathrm{Hom}_{\mathcal{C}}(a, b) \cong \mathbb{C}^{\delta_{ab}}$
- categorical version of Schur's lemma
- representation theory

Def: An object X in a fusion category $\mathcal{C}$ is simple if $\mathrm{Hom}_{\mathcal{C}}(X, X) \cong \mathbb{C}$

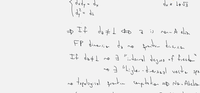
$\Rightarrow$ Morphisms between non-simple objects: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathbb{C}^n$
Intuition: trivalent vertex - basis elements of dim vector spaces

$\Rightarrow$ Consider N_{ab}^c counts from the Verlinde's fusion ring. A trivalent graph is admissibly labeled if $N_{ab}^c \neq 0$

if $N_c^{ab} \neq 0$

Ex: Fibonacci anyons $\mathbb{L} = \{1, \tau\}$, $\tau \alpha \tau = 1 + \tau$

these trivalent vertices define vertices



$\mu = 1, 2, \ldots, N_c^{ab}$

the vector space over $\mathbb{C}$

$V_c^{ab} = \mathrm{span}_{\mathbb{C}} \left\{ \left| \begin{array}{c} a \quad b \\ \diagdown \; \diagup \\ \quad c \end{array} \right\rangle_{\mu=1,\ldots,N_c^{ab}} \right\}$ $\Rightarrow$ Fusion space $\cong \mathbb{C}^{N_c^{ab}}$

$= \mathrm{Hom}(a \otimes b, c)$ $\Rightarrow$ dimension is N_c^{ab}

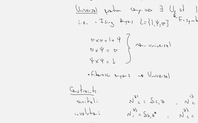
$V_{ab}^c = \mathrm{span}_{\mathbb{C}} \left\{ \left| \begin{array}{c} \quad c \\ \diagup \; \diagdown \\ a \quad b \end{array} \right\rangle_{\nu=1,\ldots,N_c^{ab}} \right\}$ $\Rightarrow$ Splitting space

$= \mathrm{Hom}(c, a \otimes b)$

$\Rightarrow \mathrm{Hom}(a, a) = \mathrm{Hom}(\mathbb{1} \otimes a, a)$

$= \mathrm{span}_{\mathbb{C}} \left\{ \left| \begin{array}{c} \mathbb{1} \quad a \\ \diagdown \; \diagup \\ \quad a \end{array} \right\rangle_{\mu=1,\ldots,N_a^{\mathbb{1}a}} \right\} \cong \mathbb{C}^1$

$\mathbb{1}$ is identity



(b isomorphism between $\mathrm{Hom}(a \otimes b, c)$ and $\mathrm{Hom}(b, c \otimes a^*)$)

$\Rightarrow$ three-fold product of simple objects

$\mathrm{Hom}(a \otimes b \otimes c, d) = \mathrm{span}_{\mathbb{C}} \left\{ \left| \begin{array}{c} a \quad b \quad c \\ \diagdown \; \diagup \; \diagup \\ \quad \alpha \quad \diagup \\ \quad \diagup \; \diagdown \\ \quad d \end{array} \right\rangle \right\}$ $\begin{array}{l} m \in \mathbb{L} \\ \alpha = 1, \ldots, N_m^{ab} \\ \beta = 1, \ldots, N_d^{mc} \end{array}$