Lecture #4

Usual notion of symmetries : 0-form, G, ... (invertible)

Generalized Symmetries

- Non-invertible symmetries (symmetries are beyond group structures)

$$D \times D = 1 + \eta$$

groups G
↓
fusion ring
$(\mathcal{L}, \{N_c^{ab}\})$

- 't Hooft anomalies
  ↳ classified $H^3(G, U(1))$
  ↳ can be naturally embedded in F-symbols
  $(\mathcal{L}, \{N_c^{ab}\}, F)$

- Higher-form symmetries
  ↳ support on sub-manifolds (e.g. strings)
  ↳ Naturally encoded in higher categories

$(n\mathcal{C})$ n-$\mathcal{C}$ ~ n category

⇝ $\underbrace{\text{Objects}}_{\text{0-morphisms}}$, $\underbrace{\text{morphisms}}_{\text{1-morphisms.}}$, 2-morphisms, ..., n-morphisms

usual categories are 1-$\mathcal{C}$

$\Rightarrow$ n-$\mathcal{C}$ $\rightsquigarrow$ p-dimensional objects correspond to p-morphisms.

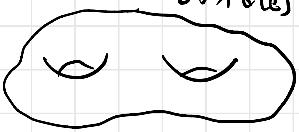
p-form symmetry operators $\rightsquigarrow$ $d-p-1$ dimensional objects.

3+1 d tori

lines

- (1-dim)  $\leftrightarrow$ 2-form symmetries

surfaces

- (2-dim)  $\leftrightarrow$ 1-form symmetries

$\Rightarrow$ $d+1$ dimensions require d-$\mathcal{C}$

- Implement braiding

$(\mathcal{C}, \{N^{ab}_c\}, F, R)$ - Braided fusion categories

(anyons)

Fusion Categories

→ Given a fusion ring $(L, \{N^{ab}_c\})$

- objects of $\mathcal{C}$ correspond to elements of fusion ring

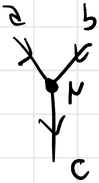
$$\mathcal{C} \ni X = \bigoplus_{i=1}^{r} n_i a_i \qquad n_i \in \mathbb{Z}_{\geq 0},\quad a_i \in L,\quad r = |L|$$

which can fuse as

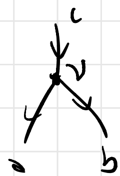
$$a \otimes b = \sum_{c \in L} N^{ab}_c \, c$$

- Morphisms $\in \mathrm{Hom}(X, Y)$ vector spaces.

focus on simple-objects $(a \in L)$

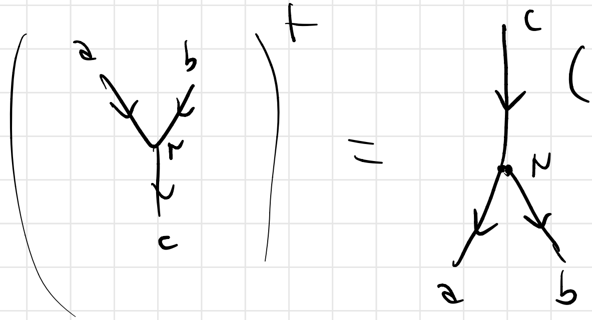
$$\mathrm{Hom}(a \otimes b, c) = V^{ab}_c = \mathrm{span}_{\mathbb{C}} \Big\{$$  $$\Big\}_{\mu = 1, \dots, N^{ab}_c}$$

dimension N^{ab}_c

$$\mathrm{Hom}(c, a \otimes b) = V^{c}_{ab} = \mathrm{span}_{\mathbb{C}} \Big\{$$  $$\Big\}_{\nu = 1, \dots, N^{ab}_c}$$

Inner product in those vector spaces

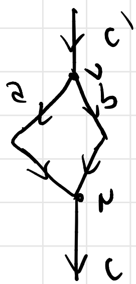
↳ Defined by the following picture



$$(\,.\,)^{\dagger} : \mathrm{Hom}(a \otimes b, c) \to \mathrm{Hom}(c, a \otimes b)$$

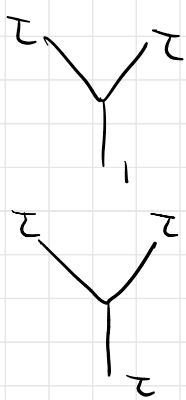
$$|\psi\rangle \to \langle\psi|$$

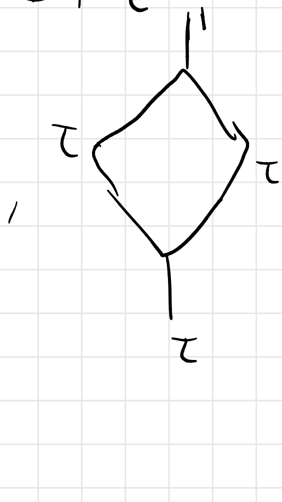
↳ Motivate us to define the inner product



$$= \delta_{\mu\nu}\, \delta_{cc'} \sqrt{\frac{d_a d_b}{d_c}}$$

$$X = \tau \times \tau = 1 + \tau$$





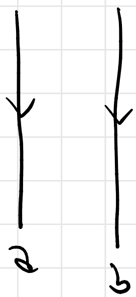
$= 0$

Completeness relation (resolution of identity)

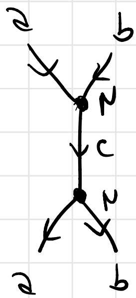
$\mathrm{Hom}(\sigma \otimes \sigma, \sigma \otimes \sigma)$

Analogous to $\sum_n |n\rangle\langle n| = \mathbb{I}$

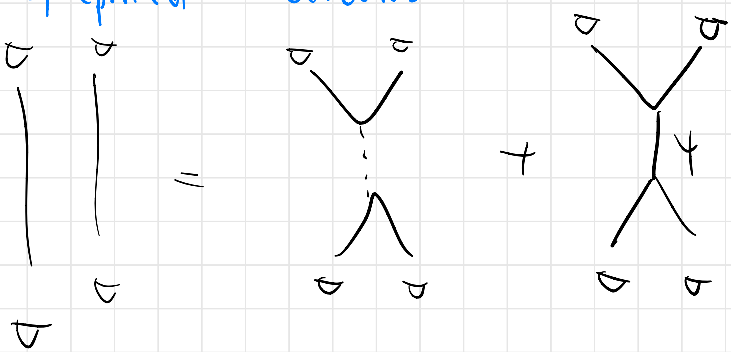
$\mathrm{Hom}(a \otimes b, a \otimes b)$



$= \sum_{c \in \mathcal{L}} \sum_{\mu=1}^{N_c^{ab}}$

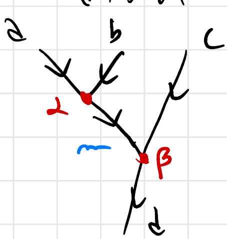


Graphical calculus.



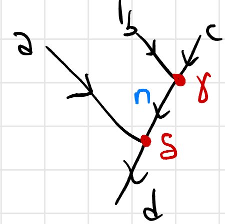
three-fold fusion products $\mathrm{Hom}(a \otimes b \otimes c, d)$

two natural way of resolving the product



$(a \otimes b) \otimes c$

or



$a \otimes (b \otimes c)$

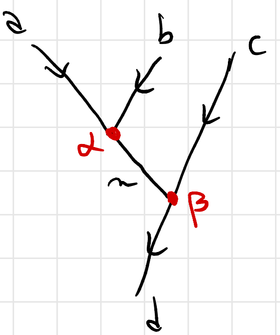
↳ two natural basis for such a vector space.

$$\mathrm{Hom}(a\otimes b\otimes c, d) = V_d^{abc}$$ with dimension

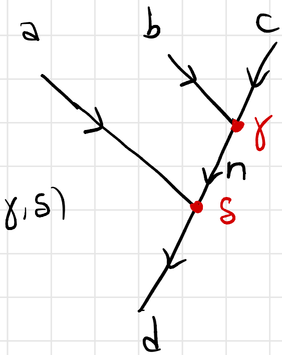
$$\dim V_d^{abc} = \sum_{m\in L} N_m^{ab}\, N_d^{mc}$$

$$= \sum_{n\in L} N_n^{bc}\, N_d^{an}$$

F-symbols: how to change from one basis into the other

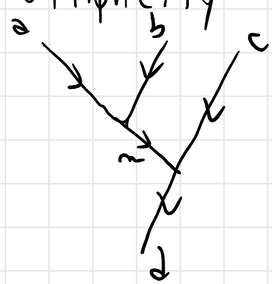


$$= \sum_n \underbrace{\left[F_d^{abc}\right]_{(m,\alpha\beta),(n,\gamma,\delta)}}_{\text{F-symbols}}$$

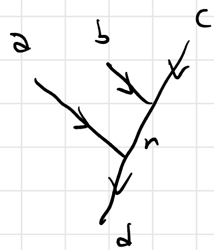


6j symbols.

Multiplicity-free fusion categories



$$= \sum_n \left[F_d^{abc}\right]_{m,n}$$



$$\text{Fusion categories} \sim \Big(\overbrace{L, \{N^{ab}_c\}}^{\text{comes from fusion ring}}, \underbrace{[F^{abc}_d]}_{\text{new input data}}\Big)$$

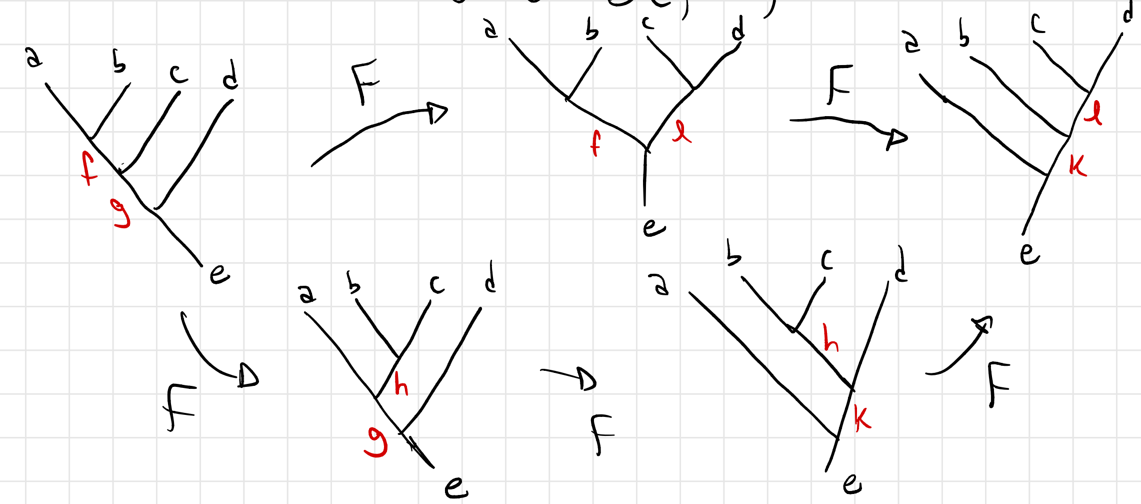
$\Rightarrow$ Unitary fusion categories $\sim$ fusion categories with unitary F-symbols.

$$[(F^{abc}_d)^{-1}]_{(n,\gamma,\delta),(m,\alpha,\beta)} = [(F^{abc}_d)^{\dagger}]_{(n,\gamma,\delta),(m,\alpha,\beta)}$$

$$= [F^{abc}_d]^{*}_{(m,\alpha,\beta),(n,\gamma,\delta)}$$

$\Rightarrow$ F-symbols must obey consistency relations.

$$\mathrm{Hom}(a \otimes b \otimes c \otimes d, e)$$



$$\left[F^{fcd}_{e}\right]_{g,l} \left[F^{abl}_{e}\right]_{f,k} =$$

$$\sum_h \left[F^{abc}_{g}\right]_{f,h} \left[F^{ahd}_{e}\right]_{g,k} \left[F^{bcd}_{k}\right]_{h,l}$$

must hold for all $a,b,c,d,e,f,g,k,l \in \mathcal{L}$.

$\hookrightarrow$ Pentagon equations

$$F \cdot F = F \cdot F \cdot F$$

- Fibonacci fusion category,

$\mathcal{L} = \{1, \tau\}$

$\tau \otimes \tau = 1 + \tau$

$$\left[F^{\tau\tau\tau}_{\tau}\right] = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{1/2} & -\phi^{-1} \end{pmatrix}, \quad \phi = \frac{1+\sqrt{5}}{2}$$

golden-ratio

- Fusion ring $\mathrm{Vec}_{\mathbb{Z}_2}$ $\rightsquigarrow$ $\mathcal{L} = \{1, g\}$

$g \otimes g = 1$

$\mathrm{Vec}\,\mathbb{Z}_2$ can be categorified to two distinct fusion categories.

$\Rightarrow$ $\underbrace{\mathrm{Vec}^{\omega_0}\,\mathbb{Z}_2}$ & $[F_g^{ggg}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\llcorner$ To boundary of Ising Ztld.

$\Rightarrow$ $\underbrace{\mathrm{Vec}^{\omega_1}\,\mathbb{Z}_2}$ & $[F_g^{ggg}] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\llcorner$ boundary of Levin Gu.

(ω_1 indicates 't Hooft anomaly.

$$Z(\mathrm{Vec}^{\omega_0}\,\mathbb{Z}_2)$$