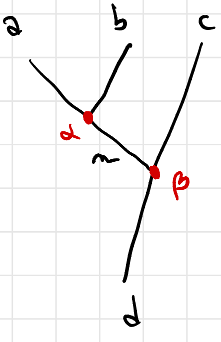
Lecture 5

Fusion rings $\xrightarrow{\text{categorify}}$ Fusion categories

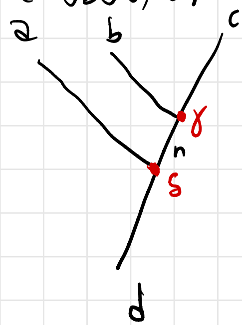
$(\mathcal{L}, \{N^{ab}_c\})$ $\qquad$ $(\mathcal{L}, \{N^{ab}_c\}, \underbrace{\{F^{abc}_d\}}_{\text{unitary matrices}})$

ring element w/ fusion $\longrightarrow$ objects morphism

$\Rightarrow$ F-symbols appear when considering a change of basis in the abstract vector space $\mathrm{Hom}(a \otimes b \otimes c, d)$



$$= \sum_{n,\gamma,\delta} [F^{abc}_d]_{(m,\alpha,\beta),(n,\gamma,\delta)}$$



Tambara-Yamagami: fusion categories

$$\mathcal{C} = TY(A, \chi, \nu)$$

A: Abelian group (finite)

$\nu = \pm 1$ Frobenius-Schur indicator

$\chi : A \times A \to U(1)$ ~ symmetric bicharacter

$$\chi(a+b, c) = \chi(a,c)\,\chi(b,c)$$
$$\chi(a, b+c) = \chi(a,b)\,\chi(a,c)$$

$L = A \cup \{m\}$ rank $|A|+1$

such that

$$g \otimes h = (g+h) \quad , g,h \in A$$

$$m \otimes a = a \otimes m = m$$

$$m \otimes m = \bigoplus_{a \in A} a$$

The only non-trivial F-symbol is

$$\left[F^{mam}_{m}\right]_{ab} = \frac{v}{\sqrt{|A|}}\,\chi^*(a,b)$$

E.g. $A = \mathbb{Z}_2$

$L = \{1, \psi, \sigma\}$

$\psi \otimes \psi = 1$

$\sigma \otimes \psi = \psi \otimes \sigma = \sigma$

$\sigma \otimes \sigma = 1 + \psi$

$\mathcal{C}_{Ising} = TY(\mathbb{Z}_2, 1, +1)$

~ Zhang + Cordova [2304.01262]

$\Rightarrow$ F-symbols by themselves are not "physical".

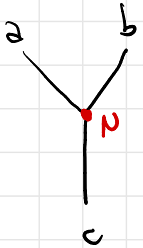
$\hookrightarrow$ There are "gauge transformations" for F-symbols.

Let $\mathcal{C} = (L, \{N^{ab}_c\}, \{F^{abc}_d\})$ and

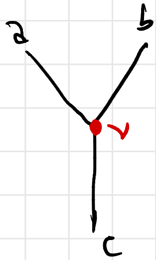
$\tilde{\mathcal{C}} = (L, \{N^{ab}_c\}, \{\widetilde{F}^{abc}_d\})$

if F and $\widetilde{F}$ are gauge equivalent, both $\mathcal{C}$ and $\tilde{\mathcal{C}}$ correspond to the same fusion category.

$\rightsquigarrow$ Consider $\mathrm{Hom}(a \otimes b, c)$



$$\longrightarrow \sum_{\nu} [\Gamma^{ab}_c]_{\mu\nu}$$



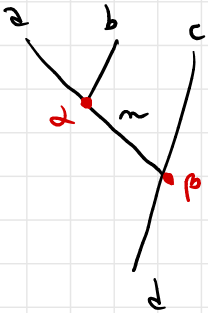
$\hookrightarrow$ We can perform a change of basis transformation

$\Gamma^{ab}_c \in GL(N^{ab}_c, \mathbb{C})$ in general

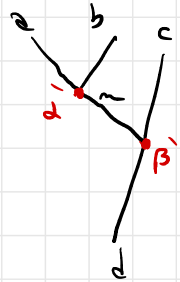
$\Gamma^{ab}_c \in U(N^{ab}_c, \mathbb{C})$ for unitary fusion categories

For mult.-free fusion cat, $\Gamma^{ab}_c \in U(1)$

$\Rightarrow$ these induce equivalence relations for vectors in $\mathrm{Hom}(a \otimes b \otimes c, d)$

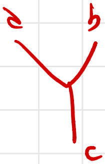


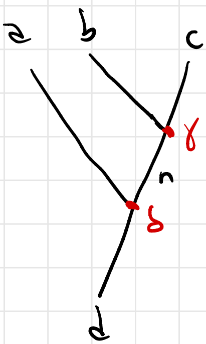
$$\longrightarrow \sum_{\alpha',\beta'} \left[\Gamma^{ab}_{m}\right]_{\alpha\alpha'} \left[\Gamma^{mc}_{d}\right]_{\beta\beta'}$$



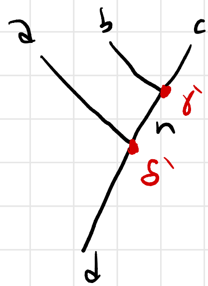
for mult.-free

Hom$(a\otimes b, c)$ has dimension 1.

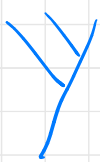
$= \mathrm{span}_{\mathbb{C}} \Big\{ e^{i\alpha}$  $\Big\}$



$$\longrightarrow \sum_{\gamma',\delta'} \left[\Gamma^{an}_{d}\right]_{\delta\delta'} \left[\Gamma^{bc}_{n}\right]_{\gamma\gamma'}$$



Comparing with

 $= F$ 

we have the identification

$$[\tilde{F}^{abc}_d]_{(m,\alpha,\beta)(n,\gamma,\delta)} = \sum_{\alpha',\beta',\gamma',\delta'} [\Gamma^{ab}_m]_{\alpha\alpha'} [\Gamma^{mc}_d]_{\beta\beta'}$$

$$\times [F^{abc}_d]_{(m,\alpha',\beta')(n',\gamma',\delta')} [\Gamma^{ab}_d]^{-1}_{\delta\delta'} [\Gamma^{bc}_n]^{-1}_{\gamma\gamma'}$$

$$\tilde{F} = \Gamma\Gamma F \Gamma^{-1}\Gamma^{-1}$$

where $\Gamma^{-1} = \Gamma^{\dagger}$ if $\mathcal{C}$ is unitary.

For multiplicity free $\mathcal{C}$

$$[F^{abc}_d]_{m,n} \to \frac{\Gamma^{ab}_m \Gamma^{mc}_d}{\Gamma^{an}_d \Gamma^{bc}_n} [F^{abc}_d]_{m,n}$$

(gauge transformations.

$\Rightarrow$ If $\{F^{abc}_d\}$ obey pentagon equation, then $\{\tilde{F}^{abc}_d\}$ (gauge equivalent to F) also obey such equations

Homework: $\{$ Pentagon $\leftrightarrow$ 3-cycle, gauge equiv $\leftrightarrow$ 3-boundaries

E.g: $\mathcal{C} = \mathrm{Vec}^{\omega}\, G$

G : finite group

$$\omega \in H^3(G, U(1)) = \{3\text{-cycles}\} / \{3\text{-coboundaries}\}$$

$\omega : G \times G \times G \to U(1)$

$L = G$, $g \otimes h = gh$, $[F_d^{abc}]_{ab,bc} = \omega(a,b,c)$

E.g.: $\mathrm{Vec}^{\omega}\mathbb{Z}_N$, $\omega \in H^3(\mathbb{Z}_N, U(1)) \simeq \mathbb{Z}_N$

$\omega \sim$ 't Hooft anomaly of G.

Gauge invariant quantities

$\rightsquigarrow \nexists$ extensive list of such quantities.

Show $e \underset{\text{gauge}}{\sim} \tilde{e}$ in general, is a hard task.

(active research topic

i) Some particular F-symbol elements.

$$[F_b^{abc}]_{b,b} \to \frac{\Gamma_b^{ab}\,\Gamma_b^{bc}}{\underbrace{\Gamma_b^{ab}\,\Gamma_b^{bc}}_{=1}}[F_b^{abc}]_{b,b}$$

(in mult-free FC) $\forall\, a,b,c \in L$.

ii) Frobenius-Schur indicator.

$$\kappa_a = \mathrm{sgn}\left([F_a^{aaa}]_{11}\right) = \pm 1 \quad \forall\, a \in L \text{ s.t. } a = a^*$$

($\rightsquigarrow [F_a^{aaa}]_{11} \in \mathbb{R} \Leftrightarrow$ isotopy of trivalent graphs.

(isotopy)

$\sim\!\Theta\!\sim = 1 + a\,\Theta \dots$

$\sim = \sim^*$

$\nu = \mathrm{sgn}\left(F^{\sim\sim\sim}_{\sim}\right)_{11}$

iii) FP are gauge inv.

iv) Clever combinations of F-symbols.

[arXiv:1410.4540] Sec. II A.

Examples

0) Ising Model in 1+1d

i) Anyonic chains: Hamiltonian models in 1+1d which are symmetric under arbitrary unitary fusion categ. $\mathcal{C}$.

ii) Kitaev Quantum Doubles $\mathcal{C} = \mathrm{Vec}\,G$ (Input)

$\downarrow$

$Z[\mathrm{Vec}\,G]$ (Output) (anyon theory)

iii) Levin-Wen string nets.

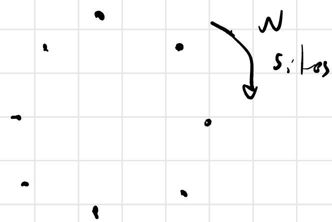
Input $\mathcal{C}$ (UFC) $\xrightarrow{Z[\mathcal{C}]}$ Output (BUFC) (anyon theory)

Ising model in 1+1 dimensions is a lattice model.

↳ tensor decomposition Hilbert space $\mathcal{H}_{total} = \bigotimes_i \mathcal{H}_i$

↳ For each site $i \rightsquigarrow \dim \mathcal{H}_i < \infty$.

on PBC
$i = 1, \dots, N$



$$H = -\sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - g \sum_{i=1}^{N} \sigma_i^x$$

$\forall$ values of g, following symmetries

- $\mathbb{Z}_2$ symmetry

$$U = \prod_{i=1}^{N} \sigma_i^x \quad , \quad U^2 = 1.$$

- $\mathbb{Z}_N$ translation symmetry

$$T = \prod_{i=1}^{N} t_i \quad , \quad T^N = 1.$$

$$t_i = \frac{1}{2}\left(\mathbb{I} + \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}\right)$$

$g > 0$

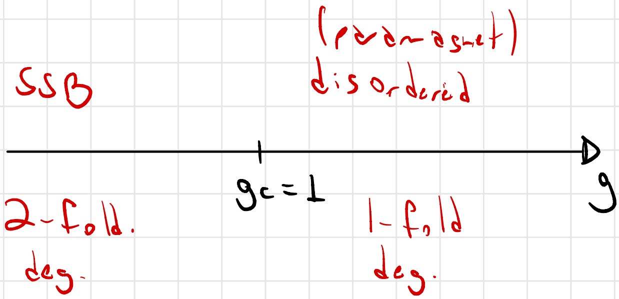
• $g \gg 1 \to \quad |GS\rangle = \bigotimes_i |+_i\rangle$ ~ 1-fold degenerate.

$$|+_i\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_i\rangle + |\downarrow_i\rangle\right) \qquad |GS\rangle = U|GS\rangle$$

• $g \ll 1 \to$

$$\left.\begin{aligned} |GS_1\rangle &= \bigotimes_i |\uparrow_i\rangle \\ |GS_2\rangle &= \bigotimes_i |\downarrow_i\rangle \end{aligned}\right\} \text{ 2-fold degenerate}$$

$|GS_2\rangle = U|GS_1\rangle$ ~ signals SSB.



⇒ Kramer Wannier D_{KW}

$$D_{KW} H = \tilde{H} D_{KW}$$

$$H = -\sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - g \sum_{i=1}^{N} \sigma_i^x \qquad \text{vs}$$

$$\tilde{H} = -g \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - \sum_{i=1}^{N} \sigma_i^x$$

For $g=1$. $[D_{uw}, H] = 0$. $\Rightarrow$ symmetry!