Lecture 6

Concrete examples 1+1 d

Ising in 1+1 d

Shu-Heng (arXiv: 2308.00747)

$$H(g) = -\sum_{j=1}^{N} Z_j Z_{j+1} - g \sum_{j=1}^{N} X_j$$ ⤳ PBC.

N = lattice sites

- $\mathbb{Z}_2 \rightsquigarrow \eta = \prod_{j=1}^{N} X_j$ spin flip
- $\mathbb{Z}_N \rightsquigarrow T = \prod_{j=1}^{N-1} t_j \;, \quad t_j = \frac{1}{2}\left(\mathbb{1} + \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}\right)$

⤷ translation

@ $g=1 \rightsquigarrow$ Extra symmetry

Consider

$$U_{KW} = e^{-\frac{2\pi i N}{8}} \left(\prod_{j=1}^{N-1} \frac{1+iX_j}{\sqrt{2}} \frac{1+iZ_jZ_{j+1}}{\sqrt{2}} \right) \frac{1+iX_N}{\sqrt{2}}$$

↳ MPO transl. inv.

$U_{KW}: \mathcal{O} \to \mathcal{O}'$ $\quad (\mathcal{O}' = U_{KW}\, \mathcal{O}\, U_{KW}^{\dagger})$

$$U_{KW}: \begin{cases} X_j \to Z_j Z_{j+1} \\ Z_j Z_{j+1} \to X_{j+1} \end{cases} \quad j = 1, \dots, N-1.$$

for $j = N$, however

$$U_{KW}: \begin{cases} X_N \to \eta\, Z_N Z_1 \\ Z_N Z_1 \to \eta\, X_1 \end{cases}$$

Non-local operators

$\to$ U_{KW} – not a symmetry $\quad ([U_{KW}, H] \neq 0)$.

Use projector into the even $\mathbb{Z}_2$ sector ($\eta = 1$) and define

$$D = \sqrt{2}\, U_{KW} \left(\frac{1+\eta}{2}\right), \text{ translation invariant.}$$

$[D, H] = 0$.

– maps local operators into local operators

Kramers-Wannier duality

$\to$ maps small g into large g

SSB $\qquad\qquad$ disordered.

KW: $\quad H(g) \longrightarrow g\, H(1/g)$

Modern understanding: dualities correspond to different (equivalent) descriptions of same physics.

$$D\, H(g) = g\, H(1/g)\, D$$

$\Rightarrow$ KW is a duality in the $\mathbb{Z}_2$ even subspace of total Hilbert space.

@ $g=1$ $\quad D$ is a symmetry

Aside, $g \ll 1$

$|\uparrow\uparrow\ldots\uparrow\rangle, \quad |\downarrow\downarrow\ldots\downarrow\rangle$

$$|\text{even}\rangle = \frac{|\uparrow\uparrow\ldots\uparrow\rangle + |\downarrow\downarrow\ldots\downarrow\rangle}{\sqrt{2}}$$

$|\uparrow\uparrow\uparrow\ldots\uparrow\rangle = \eta\, |\downarrow\downarrow\downarrow\ldots\downarrow\rangle$ $\quad\Rightarrow\quad$ $|\text{odd}\rangle = \dfrac{|\uparrow\uparrow\ldots\uparrow\rangle - |\downarrow\downarrow\ldots\downarrow\rangle}{\sqrt{2}}$

$\hookrightarrow$ SSB! η is NOT a symmetry of ground state space.

$P\,|\text{even}\rangle = |\text{even}\rangle$

$P\,|\text{odd}\rangle = 0.$

$P = \dfrac{1+\eta}{2}$ $\qquad |\text{odd}\rangle \in \ker D$

$$P|\mathrm{odd}\rangle = \frac{1}{2}|\mathrm{odd}\rangle + \frac{\eta}{2}|\mathrm{odd}\rangle = \frac{1}{2}|\mathrm{odd}\rangle - \frac{1}{2}|\mathrm{odd}\rangle$$

$$= 0.$$

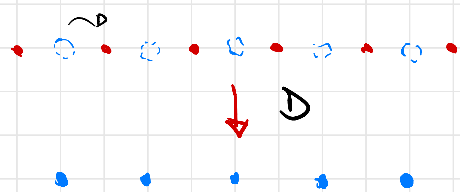
$$\boxed{P_\eta = P.}$$

Symmetry structure $S = \{\eta, T, D\}$

$$\eta^2 = 1, \quad T^N = 1, \quad TD = DT, \quad T\eta = \eta T$$

$$D\eta = \eta D = D$$

$$D^2 = (1+\eta)T$$



$\Rightarrow$ D ~ half-translation

thermodynamic limit $N \to \infty$ $\Rightarrow$ S is not a fusion category (S is not finite, D^2 does not decompose into sum of simple object, etc...)

↳ Seiberg + Shu-Heng (?)

If we can get rid of translations, we have a honest fusion category symmetry

@ $g=1$ ⇝ Ising CFT

CFT 101 [ArXiv: 9108028]

Field theories symmetric under conformal symmetries

- translations, rotations, Lorentz (Poincare group)
- Scaling
- Special conformal transformations.

Conformal invariance ⇝ Angle preserving transformations.

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega(x)\, g_{\mu\nu}$$

In 1+1 d a CFT can be specified by a minimal set of data:

i) Spectrum of primary $\quad \mathcal{O}_i \sim \overbrace{(h_i, \bar{h}_i)}^{\text{weights.}}$

$$\Delta_i = h_i + \bar{h}_i \quad \text{scaling dim}$$

$$s_i = h_i - \bar{h}_i \quad \text{spin}$$

ii) central charge c.

(operator product expansion)

iii) OPE coefficient

$$\mathcal{O}_i(z)\,\mathcal{O}_j(0) = \sum_k N^{ij}_k \, z^{h_k - h_i - h_j} \, \bar{z}^{\bar{h}_k - \bar{h}_i - \bar{h}_j} \, \mathcal{O}_k(0)$$

iv) Modular invariant characters $\chi_i(\tau)$, $\bar{\chi}_i(\bar{\tau})$

→ determine modular partition function on a torus

v) More data } Lie Group
Supersymmetry
⋮

<u>Ex:</u> Ising CFT

				$(h, \bar{h})$
i)	$1(x)$	(identity)	$\sim$	$(0,0)$
	$\varepsilon(x)$	(thermal)		$(1/2, 1/2)$
	$\sigma(x)$	(spin)		$(1/16, 1/16)$

local quantities

ii) $c = 1/2$

iii) OPE fusion rules

$$\varepsilon \times \varepsilon = 1$$

$$\sigma \times \varepsilon = \varepsilon \times \sigma = \sigma$$

$$\sigma \times \sigma = 1 + \varepsilon$$

iv) Mod. inv. (di Francesco)

Def: A CFT is called diagonal if all primaries have vanishing spin

$$s_i = 0 \Rightarrow h_i = \bar{h}_i$$

$$\Rightarrow \mathcal{O}_i \rightsquigarrow (h_i, h_i)$$

$\Rightarrow$ For diagonal CFTs, every primary field defines a topological line operator (Verlinde lines)

$$\mathcal{O}_i \Leftrightarrow \mathcal{D}_i$$

$$\mathcal{O}_i \times \mathcal{O}_j \sim \sum_k N^{ij}_k \, \mathcal{O}_k \iff D_i \times D_j = \sum N^{ij}_k D_k$$

Eg: Ising.

3 primaries $\quad 1 \,, \quad \varepsilon \,, \quad \sigma$

3 topological lines $\quad 1 \,, \quad \eta \,, \quad D$

$$\varepsilon \times \varepsilon = 1 \qquad\qquad \eta \times \eta = 1$$

$$\sigma \times \sigma = 1 + \varepsilon \quad \Longleftrightarrow \quad D \times D = 1 + \eta$$

$$\sigma \times \varepsilon = \sigma \qquad\qquad D \times \eta = D.$$

$\Rightarrow$ the set of (Verlinde) topological lines are symmetries of the CFT

$$[D^i, T] = 0$$

stress-tensor / energy-momentum tensor.

and form a fusion category $\mathcal{C}$ with F-symbols can be derived from CFT data.

$\Rightarrow$ Moore + Seiberg: Lectures on rational CFTs.

Ising fusion category $\mathcal{C}$ specified by

$$1, \ \eta, \ D$$

w/ fusion rules

$$\eta \times \eta = 1$$

$$D \times \eta = D$$

$$D \times D = 1 + \eta$$

and appropriate F-symbols.

Anyon chains (Aasen - Fendley - Mong)

Input: $\mathcal{C} = \{L, \{N_c^{ab}\}, \{F_d^{abc}\}\}$

$$\downarrow$$

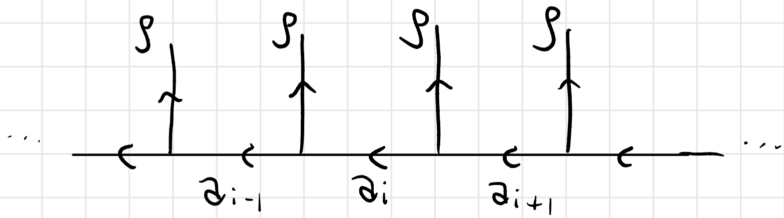
Output: Hamiltonians that realize $\mathcal{C}$ fusion category symmetry.

$\Rightarrow$ they are not quite lattice models

$$\mathcal{H}_{\text{total}} \neq \bigotimes_i \mathcal{H}_i$$

Anyon chains: Fix a fusion category $\mathcal{C}$ (mult.-free for simplicity) and fix a simple object $\rho \in \mathcal{C}$

We consider chains:



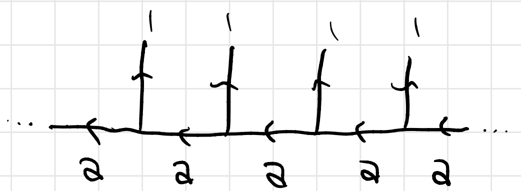
$a_i \in \mathcal{C}$ simple objects.

$\mathcal{H} \neq \otimes \mathcal{H}_i$

Hilbert space

$$\mathcal{H}_{total} = \mathrm{span}_{\mathbb{C}} \left\{ |a_1 a_2 \dots a_N\rangle \;\middle|\; \overbrace{\text{admissible labeled}} \; a_i \in L \right\}$$

Ex: $\mathcal{C}_{Ising}$ on $L = \{1, \eta, D\}$. and fix reference object $\rho = 1$.

$\mathcal{H}_{total} = \mathrm{span}_{\mathbb{C}} \Big\{$  $, \; a \in L \Big\}$

$$\Rightarrow \dim \mathcal{H}_{total} = 3 \neq \bigotimes_i \mathcal{H}_i$$

- choose $\rho = \eta$ $\Rightarrow$ $\dim \mathcal{H}_{total} = 3$

- choose $\rho = D$ $\Rightarrow$ $\dim \mathcal{H}_{total} = 2^N$

[arXiv: 2305.05774]