Lecture 7

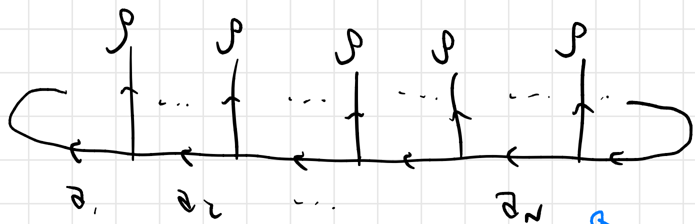
Anyon chain

Input $\mathcal{C}, \rho$ ← simple reference object

$$\mathcal{H}, \hat{H} \longrightarrow$$ 1+1 d quantum theory w/ symmetry $\mathcal{C}$

($\mathcal{H}$: Hilb space, $\hat{H}$: Hamiltonian)

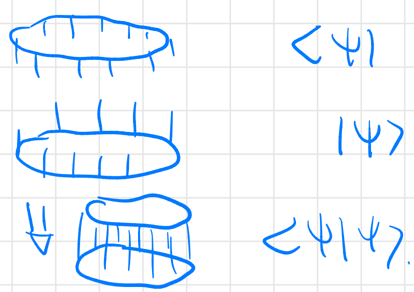
States in $\mathcal{H}$ correspond to admissibly labelled graphs



✓ if $a_{j+1} \in a_j \otimes \rho$

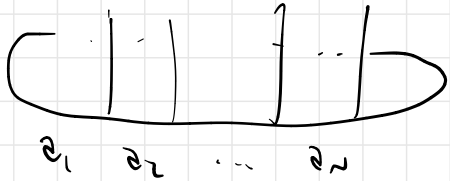
← simple objects in $\mathcal{C}$

$\Rightarrow$ Many body Hilb. space requires ρ to be non-invertible.

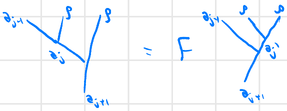


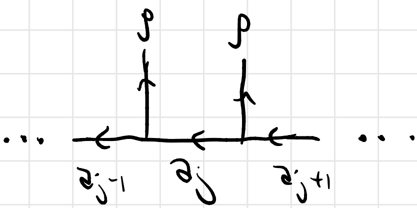
ket notation

$$|a_1 a_2 \dots a_N\rangle =$$

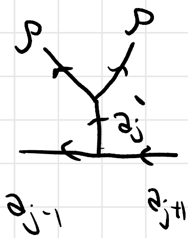


$$\mathcal{H} = \mathrm{span}_{\mathbb{C}} \left\{ \left. |a_1 a_2 \dots a_N\rangle \right|_{\text{admissibly labelled}} \right\}$$





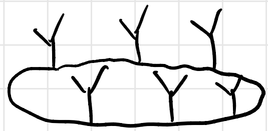
$$= \sum_{a_j' \in L} \left[F^{a_{j-1}\rho\rho}_{a_{j+1}} \right]_{a_j a_j'}$$



from ket notation

$$|\dots a_{j-1}\, a_j\, a_{j+1} \dots\rangle = \sum_{a_j' \in L} \left[F^{a_{j-1}\rho\rho}_{a_{j+1}} \right]_{a_j a_j'} |\dots a_{j-1}\, a_j'\, a_{j+1} \dots\rangle$$

$\Longleftrightarrow$



N vertical segments

$\to \frac{N}{2}$ vertical segments

Hamiltonian

For each simple $a \in L$ and a lattice site $1 \leq j \leq N$, we define the projector

$$P_j^a \,|\cdots a_{j-1}\, a_j\, a_{j+1}\rangle^\vee = -\delta_{a_j, a}\, |\cdots a_{j-1}\, a_j\, a_{j+1} \cdots\rangle^\vee$$

satisfy $\displaystyle\sum_a P_j^a = -1$

$$\boxed{\begin{gathered} P_n = |n\rangle\langle n| \\ \sum_n P_n = 1 \end{gathered}}$$

$$P_j^a P_j^a = -P_j^a$$

(optional Homework)

We define the Hamiltonian $\quad \| P_j^a \| \sim O(1)$

$$H = \sum_a A(a) \sum_j P_j^a$$

Hamiltonian parameter

↳ translation $\mathbb{Z}_N$ symmetry

H has $\mathcal{L}$ symmetry!

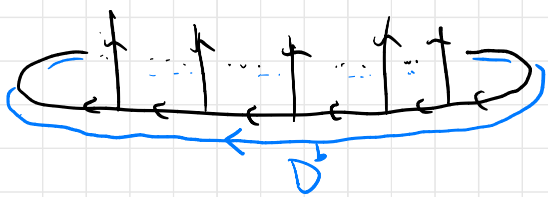
$\rightsquigarrow$ for $a \in \mathcal{L}$ we define D^a which acts on $\mathcal{H}$.

- $[D^a, H] = 0 \quad \forall\, a \in \mathcal{L}$

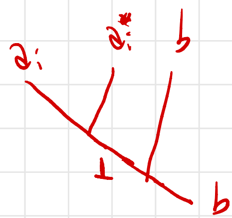
  (to follow from pentagon equation.)

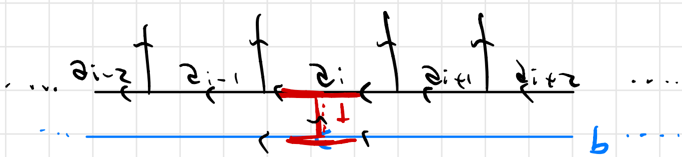
- $$D^a D^b = \sum_c N^{ab}_c \, D^c$$

Action of D^b over a state is given by:

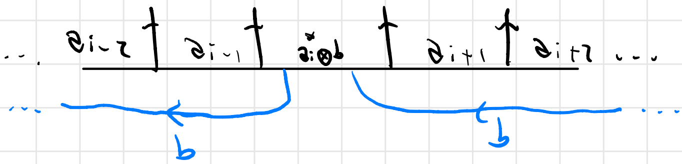


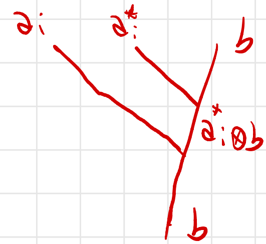
We resolve this through F moves:



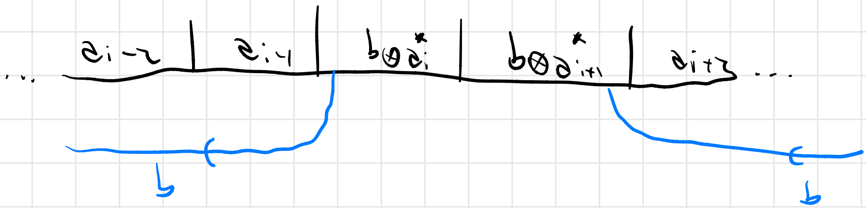


$\downarrow$ F-move $\left[F^{a_i a_i^* b}_{b}\right]_{1,\, b\otimes a_i}$





$$\downarrow \quad \left[F^{a_{i+1}\, a_{i+1}\, b}_{b} \right]_{1,\, b \otimes a_{i+1}}$$



for each $b \in L$:

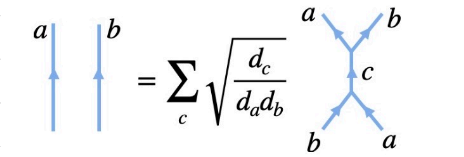
$$D^b |a_1 \dots a_N\rangle = \sum_{\{\dots\}} \underbrace{F F F \dots}_{N \text{ times}} |\tilde{a}_1 \dots \tilde{a}_N\rangle$$

Sec 1.2.

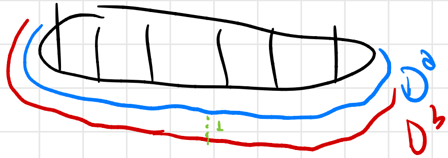
[arXiv: 2305.05774] Inamura + Ohmori

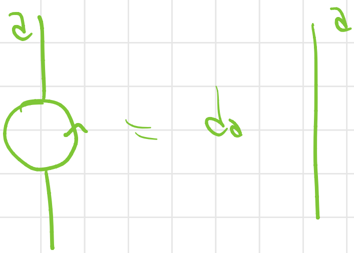
(b $\rho \in \mathcal{C}$ (not necessarily simple))

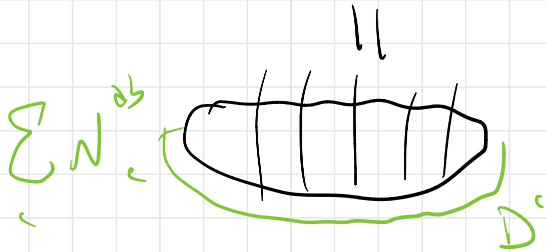
Aside:



Hint:

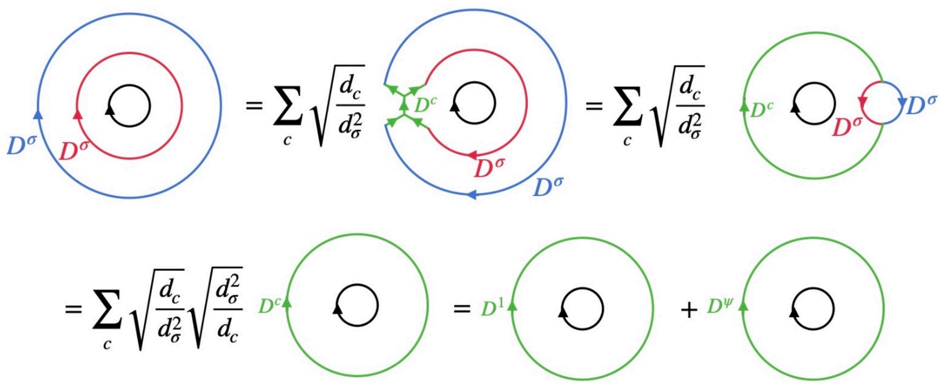






See Lecture #4 for more.

bubble-popping move to simplify such expression



Ex: • $\mathcal{C} = \mathrm{Vec}_G$, $\rho = \bigoplus_{g \in G} g$

$\rightsquigarrow$ G-symmetric spin chain

$\mathcal{H} = \bigotimes_i \mathcal{H}_i$

• $\mathcal{C} = \mathrm{Rep}(G)$, $\rho \in \mathrm{Rep}(G)$

$\hookrightarrow$ Gauged version G-symmetric spin chain

$\hookrightarrow$ boundary theories of quantum double $D(G)$

[2211.12096] ~ Alice + others.

• $\mathcal{C} = \mathrm{Fib}$ $\quad 1, \tau, \quad \tau \otimes \tau = 1 + \tau$

[arXiv:0612341] Feiguin + Kitaev + others.

$\hookrightarrow$ "Golden chain"

critical state $\rightsquigarrow$ $c = \frac{7}{10}$ tricritical Ising CFT

• $\mathcal{C} = \mathrm{Ising}$ $\quad 1, \psi, \sigma$.

σ TASI Shu-Heng.

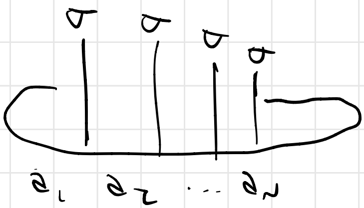
$$\text{Ising} = TY_+(\mathbb{Z}_2) \quad \mathcal{L} = \{1, \psi, \sigma\}$$

w/ ref. object σ.

$$F^{\sigma\sigma\sigma}_{\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad [F^{\psi\sigma\psi}_{\sigma}]_{\sigma\sigma} = -1$$

$$[F^{\sigma\psi\sigma}_{\psi}]_{\sigma\sigma} = 1.$$

Chain of size N

$\mathcal{H} = \Big\{$ 

$\Big|$ admissibly labelled $\Big\}$

If N odd $\Rightarrow$ $\mathcal{H} = \emptyset$

If N even $\mathcal{H} = \mathcal{H}_{\text{even}} \oplus \mathcal{H}_{\text{odd}}$

$\mathcal{H}_{\text{odd}} = \text{Span}_{\mathbb{C}} \Big\{$ 

$\Big\}$

$\mathcal{H}_{\text{even}} = \text{Span}_{\mathbb{C}} \Big\{$ 

$\Big\}$

$\mathcal{H}_{even}$ and $\mathcal{H}_{odd}$ have dim $2^{\frac{N}{2}}$

$\mathcal{H}_{total} = \mathcal{H}_{even} \oplus \mathcal{H}_{odd}$ has dim $2^{\frac{N}{2}+1}$

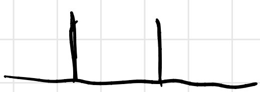
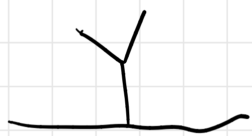
$\neq \bigotimes \mathcal{H}_i$

Usual Ising model w/ $N/2$ sites has dim $2^{N/2}$

Let us focus $\mathcal{H}_{odd}$ $\quad a_j = \begin{cases} \sigma & j \text{ odd} \\ 1, \psi & j \text{ even} \end{cases}$

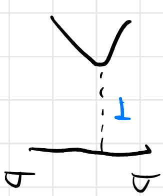
Denote

$|0\rangle_j \leftrightarrow a_j = 1 \quad , \quad |1\rangle_j \leftrightarrow a_j = \psi$

 $= \sum F$ 

z-basis $\qquad\qquad$ x-basis

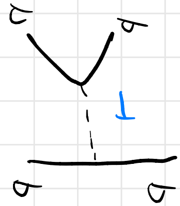
Ex:

$= \frac{1}{\sqrt{2}}$ 

$+ \frac{1}{\sqrt{2}}$ 

$|0\rangle \qquad |+\rangle \qquad |-\rangle$

$|1\rangle$ $= \frac{1}{\sqrt{2}}$



$|+\rangle$ $- \frac{1}{\sqrt{2}}$

$|-\rangle$.

↳ two term

$$P_j^1 \quad , \quad P_j^\psi$$

However $P_j^1 + P_j^\psi = -1 . \quad \forall j.$

Let us focus on P_j^1 ~ acts differently if j is even or odd.

for j odd:

$$P_j^1 |\dots \sigma a_j \sigma \dots\rangle = -\delta_{a_j,1} |\dots \sigma a_j \sigma \dots\rangle$$

↳ favors $a_j = 1$ ~ $|+\rangle_j$

disfavors $a_j = \psi$ ~ $|-\rangle_j$

$$P_j^1 = -\left(\frac{1+\sigma_j^x}{2}\right)$$

Similarly for j even

P_j^1 for j even:

favor equal states in z basis

$$P_j^1 = -\left(\frac{1+\sigma_{j-1}^z\sigma_{j+1}^z}{2}\right)$$

$$H = -\alpha \sum_{j\,\mathrm{odd}} \left(X_j + Z_j Z_{j+2} \right) + \mathrm{etc.}$$

A similar analysis on H_{even}

$$H = -\alpha \sum_{j\,\mathrm{even}} \left(X_j + Z_j Z_{j+2} \right) + \mathrm{etc.}$$

$\Rightarrow$ Action $\mathcal{H}_{total}$

$$H = -\alpha \sum_j \left(X_j + Z_j Z_{j+2} \right)$$

$\hookrightarrow$ $c = \frac{1}{2}$ Ising CFT

D^1 , D^ψ , D^σ

D^ψ acts within each $\mathcal{H}_{even}$ or $\mathcal{H}_{odd}$ subspace

D^σ acts within the full $\mathcal{H}_{total}$

traverse Ising	Anyon chain $\mathcal{C} = TY_+(\mathbb{Z}_2)$
$\mathcal{H} = \otimes \mathcal{H}_i$	$\mathcal{H} = \mathcal{H}_{odd} \oplus \mathcal{H}_{even}$
$D^2 = (1+\eta)\, T$	$D^\sigma \cdot D^\sigma = 1 + D^\psi$