Lecture 8

Fusion Cat. $\to$ Anyonic chains

Mon Braided " " $\to$ Levin-Wen   Lucas : April 6
arXiv: 0404617

Mon TQFTs   Quantum Double   Bruno : April 10
arXiv: 9707021

Braided fusion categories

Recap: fusion category

$$\mathcal{C} = (L, N^{ab}_c, F^{abc}_d)$$

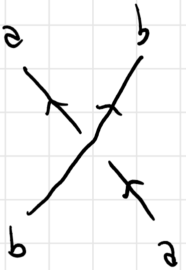
w/ L, N^{ab}_c satisfy fusion ring axioms and F symbols satisfy pentagon equation.

Now: we equip fusion categories w/ a braiding structure.

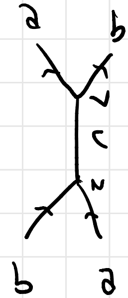
Def: A braided fusion category $\mathcal{C}$ is a fusion category with braiding isomorphisms in

$$\mathrm{Hom}(a \otimes b, a \otimes b)$$

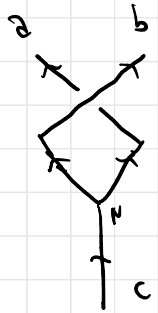
defined by the following properties



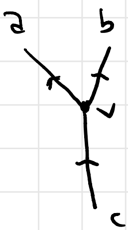
$$= \sum_{c,\mu,\nu} \sqrt{\frac{d_c}{d_a d_b}} \left[R^{ab}_c\right]_{\mu\nu}$$

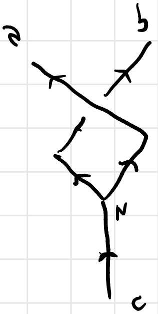


$\Rightarrow$ $[R^{ab}_c]$ are $N^{ab}_c \times N^{ab}_c$ unitary matrices (for unitary braided fusion categories)

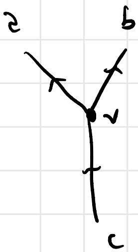


$$= \sum_{\nu} \left[R^{ab}_c\right]_{\mu\nu}$$





$$= \sum_{\nu} \left[(R^{ab}_c)^{-1}\right]_{\mu\nu}$$

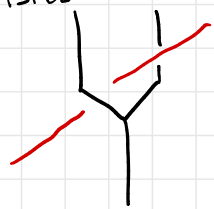


[arXiv:1410.4540]

they obey the following relations

and similarly

Aside



↳ requires R-move

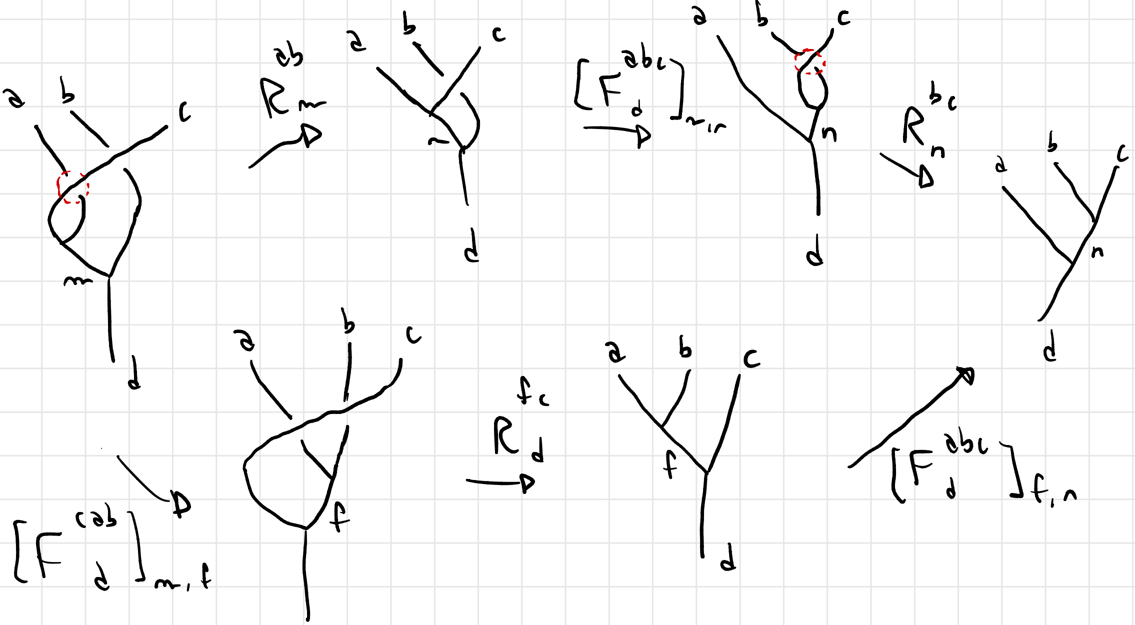
↳ Yang-Baxter equations for braiding operators.

~ lines can slide over braids / vertices.

R-symbols must satisfy hexagon equations for consistency of category!

⇝ Vectors in Hom $(a \otimes b \otimes c, d)$

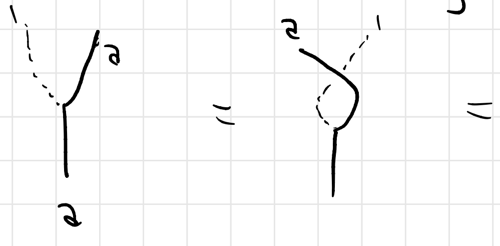
Multiplicity-free case, for simplicity:



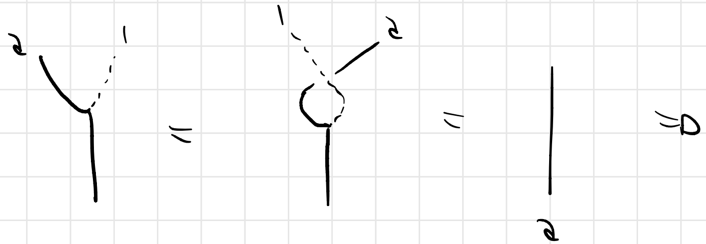
$$R^{ab}_m \, [F^{abc}_d]_{m,n} \, R^{bc}_n = \sum_f [F^{cab}_d]_{m,f} \, [R^{fc}_d] \, [F^{abc}_d]_{f,n}$$

↳ $RFR = FRF$

Constraints involvolving the trivial object $1 \in \mathcal{L}$



$\Rightarrow R^{1a}_a = 1$



$\Rightarrow R^{a1}_a = 1$

$\forall a \in \mathcal{L}$.

$$\left[\left(R^{ab}_c\right)^{-1}\right]_{\mu\nu} = \left[\left(R^{ab}_c\right)^{*}\right]_{\nu\mu}$$

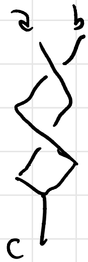
$\Rightarrow$ Unitary Braided Fusion Category:

$$\mathcal{C} = \left(L,\ N^{ab}_c,\ [F^{abc}_d],\ [R^{ab}_c]\right)$$

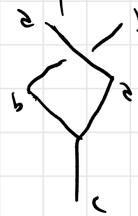
Some braiding are trivial $\Big\{$ bosonic / fermionic statistics $\Big\}$

Define symmetric center of $\mathcal{C}$

$$Z_2(\mathcal{C}) = \{ a \in L \mid R^{ab}_c \cdot R^{ba}_c = 1 \quad \forall\, b, c \in L \}$$



$= (R^{ab}_c)^{-1}$



$\ast\ R^{ab}_c = (R^{ba}_c)^{-1}$

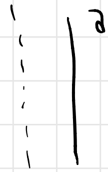
$$= \underbrace{(R^{ab}_c)^{-1} (R^{ba}_c)^{-1}}_{1}$$

$\boxed{\ast \text{ To check}}$

Elements of $Z_2(\mathcal{C})$ are called "trasparents"

$\Rightarrow$ E.g.: trivial element $1 \in L$ is trasparent.

$=$



If $\exists\ a \in L$ such that $a \in Z_2(\mathcal{C})$ and $a \neq 1 \Rightarrow \mathcal{C}$ is degenerate.

Otherwise $\mathcal{C}$ is non-degerate

Ex: Toric code $1, m, e, \psi$

$Z_2(\mathcal{C}) = \{1\}$.

Modular data

$$S_{ab} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix}_{N \times N}$$

Verlinde's formula.

for degenerate $\mathcal{C}$ $\Rightarrow$ S_{ab} is non-invertible
↳ non L.I rows / columns.

Anyon theories (topological order) are classified by Unitary $\underbrace{\text{Modular}}_{\text{non-degenerate}}$ fusion categories

↳ UMTCs = UMFCs

tensor

Remarks:

① Not every fusion category can be equipped with a braiding structure

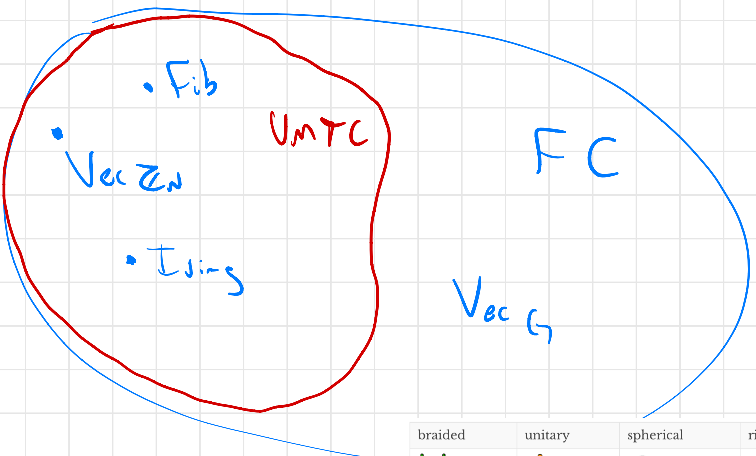
E.g.: Vec G for non-Abelian G.

$\rightsquigarrow$ Non-commutativity of fusion rules

$$h \otimes g \neq g \otimes h$$

is an obstruction to braiding.

E.g.: Haagerup $X \otimes w = Z \neq w \otimes X = Y$



braided	unitary	spherical	ribbon	modular

Anyon Wiki:

Show 50 entries Search:

Formal Name	Name	Rank	$\mathcal{D}^2_{FP}$	Data					
$FC^{1,1,0}_{1,1,1,1}$	$[\text{Trivial}]^1_{1,1}$	1	1.	data	✓	✓	✓	✓	✓
$FC^{2,1,0}_{1,1,1,1}$	$[\mathbb{Z}_2]^1_{1,1}$	2	2.	data	✓	✓	✓	✓	✓
$FC^{2,1,0}_{1,1,1,2}$	$[\mathbb{Z}_2]^2_{1,1}$	2	2.	data	✓		✓	✓	✓
$FC^{2,1,0}_{1,1,2,1}$	$[\mathbb{Z}_2]^1_{1,2}$	2	2.	data	✓	✓	✓	✓	✓
$FC^{2,1,0}_{1,1,2,2}$	$[\mathbb{Z}_2]^2_{1,2}$	2	2.	data	✓		✓	✓	✓
$FC^{2,1,0}_{1,2,1,1}$	$[\mathbb{Z}_2]^1_{2,1}$	2	2.	data	✓	✓	✓	✓	
$FC^{2,1,0}_{1,2,1,2}$	$[\mathbb{Z}_2]^2_{2,1}$	2	2.	data	✓		✓	✓	

② A fixed fusion category may admit multiple (inequivalent) braiding structures

E.g.

$$
\begin{tikzcd}
 & \mathrm{Rep}(\mathbb{Z}_2) = \{1, e\} \quad e \times e = 1 \\
\mathrm{Vec}\,\mathbb{Z}_2 \arrow[ur, "\text{symmetric braiding}"] \arrow[dr, "\text{non-degenerate braiding}"'] & \\
 & \mathrm{Sem} = \{1, s\}, \quad s \times s = 1.
\end{tikzcd}
$$

③ there is a systematic way of, given a fusion category $\mathcal{C}$, producing a braided one

$$\mathcal{Z}(\mathcal{C}) \text{ - Drinfeld center of } \mathcal{C}$$

braided fusion

fusion

$$
\begin{tikzcd}
\boxed{\text{UFC } \mathcal{C}} \arrow[r, "\text{Drinfeld Center}"] \arrow[dr, "\text{Levin-Wen String Net}"'] & \boxed{\text{UMTC } \mathcal{Z}(\mathcal{C})} \arrow[d, "\text{Algebraic theory of anyons}"] \\
 & \boxed{\text{top. ordered phase of matter}}
\end{tikzcd}
$$

④ Not every UMTC comes from $\mathcal{Z}(\mathcal{C})$ for some $\mathcal{C}$.

- $\mathcal{Z}(\mathcal{C})$ can only produce non-chiral theories $c_- = 0$ (chiral central charge)

  FQHE are chiral

- $\mathcal{Z}(\mathcal{C})$ cannot produce all theories w/ $c_- = 0$

  $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_2}) \sim$ toric code anyon theory.

  $\mathrm{Fib} \neq \mathcal{Z}(\mathcal{C})$ for some $\mathcal{C}$.

⑤ If $\mathcal{C}$ and $\mathcal{D}$ are fusion categories and

$$\mathcal{Z}(\mathcal{C}) \simeq \mathcal{Z}(\mathcal{D})$$

then $\mathcal{C}$ and $\mathcal{D}$ are said to be Morita equivalent.

Eg: Vec_G and $\mathrm{Rep}\,G$ are Morita equivalent.

⑥ There exists only finitely many UMTCs with a fixed # of simple objects.

↳ Allow us to classify all top. orders

braided	unitary	spherical	ribbon	modular

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$\mathrm{FC}^{2,1,0}_{1,1,1,2}$	$[\mathbb{Z}_2]^2_{1,1}$	2	2.	data		?			
$\mathrm{FC}^{2,1,0}_{1,1,2,1}$	$[\mathbb{Z}_2]^1_{1,2}$	2	2.	data					
$\mathrm{FC}^{2,1,0}_{1,1,2,2}$	$[\mathbb{Z}_2]^2_{1,2}$	2	2.	data		?			
$\mathrm{FC}^{2,1,0}_{1,2,1,1}$	$[\mathbb{Z}_2]^1_{2,1}$	2	2.	data					
$\mathrm{FC}^{2,1,0}_{1,2,1,2}$	$[\mathbb{Z}_2]^2_{2,1}$	2	2.	data					

$\rightsquigarrow$ Sem $\{1, s\}$

$\to$ $\overline{\text{Sem}}$ $\{1, s\}$

$\rightsquigarrow$ Rep $\mathbb{Z}_2$ $\{1, e\}$

$\Rightarrow$ All UMTC w/ rank up to 12 are completely classified

↑

Xiao-Gang Wen + friends.

Next: Modular data $\rightsquigarrow$ d_a, S_{ab}, T_{ab}, c_-, ...

+

Drinfeld center, e.g. $Z(\mathrm{Vec}_G)$