Lecture 9 :

Unitary Braided fusion categories

$$\mathcal{C} = \Big(\mathcal{L},\ N^{ab}_c,\ \underbrace{F^{abc}_d,\ R^{ab}_c}_{\text{Unitary matrices}}\Big)$$

$|\mathcal{L}| =$ rank $\mathcal{C}$

$\mathcal{C} \sim$ Topological order

(Wen, Kong, etc.)

1-form symmetries
$\hookrightarrow$ gauging !
$\Leftrightarrow$ Capelini

Analogous to F-symbols, R-symbols are not gauge invariant

Given two UBFC

$\mathcal{C} \leadsto \bar{\mathcal{C}}$ Verifying wether $\mathcal{C}$ and $\bar{\mathcal{C}}$ are gauge equivalent is a hard task. $\sim \mathcal{O}\left(e^{|\mathcal{L}|}\right)$.

Gauge invariant quantities: <u>Modular data</u>

Spoiler: Modular data does not exhaust the list of gauge invariant quantities in UBFC.

↳ Smallest example for this has rank 49.

Mod. data $\mathcal{C}_{49}$ = Mod. data $\overline{\mathcal{C}_{49}}$

but $\mathcal{C}_{49} \not\leftrightarrow \overline{\mathcal{C}_{49}}$

Not gauge equivalent.

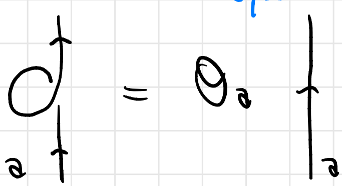
Modular data: T and S matrices.

Consider UBFC $\mathcal{C} = (\mathcal{L}, N^{ab}_c, F^{abc}_d, R^{ab}_c)$

- for each $a \in \mathcal{L}$ we define $d_a \in \mathbb{R}_{>0}$: quantum dimension (computed from N^{ab}_c, measures "non-Abelianess" of a).

- for each $a \in \mathcal{L}$ we define $\theta_a \in U(1)$: topological spin of a.
(computed from R-symbols, measures self-braiding)

Defined



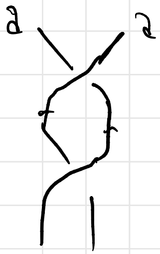
⤳ twist

⇝ Non-Abelian anyon $d_a \neq 1$ ⇝ unitary matrices capture operations in this space

⇝ $\theta_a = \frac{1}{d_a} \sum_c d_c \, \mathrm{Tr}\left[R^{aa}_c\right]$ ArXiv:1506.05805

↑ trace over $\mu, \nu = 1, \dots, N^{aa}_c$

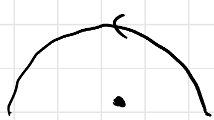
↑ Homework.



a: non-Abelian anyons.

$(R^{aa})^2 \sim$ braiding matrix

$$\psi'(x_2, x_1) = U \psi(x_1, x_2)$$



$|\theta_a| = 1$

$$\begin{cases} \theta_a = 1 \sim \text{boson} \Leftrightarrow s_a = 0 \\ \theta_a = -1 \sim \text{fermion} \Leftrightarrow s_a = 1/2 \\ \theta_a = e^{i\pi/2} \sim \text{semion} \Leftrightarrow s_a = 1/4. \end{cases}$$

$\theta_a = e^{2\pi i s_a}$, $s_a = \arg \frac{\theta_a}{2\pi} \mod 1$

(topological spin.

for general sa ~> anyon

↳ "any" top. spin

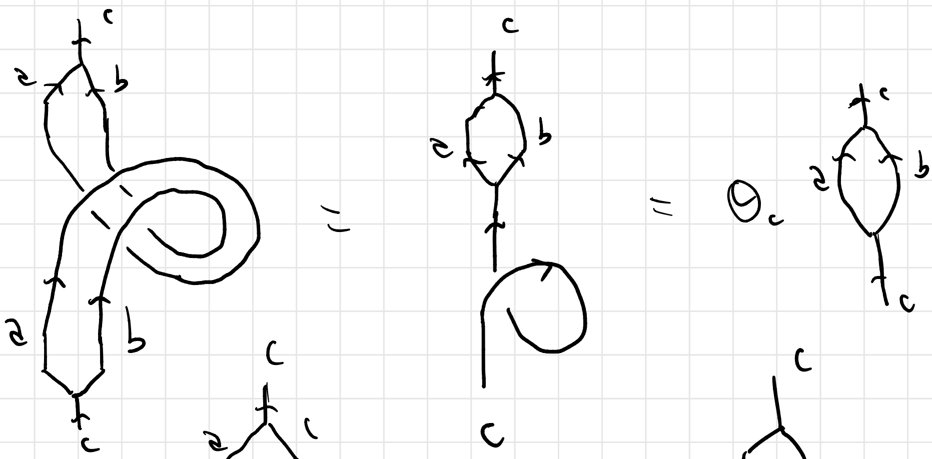
to be checked.

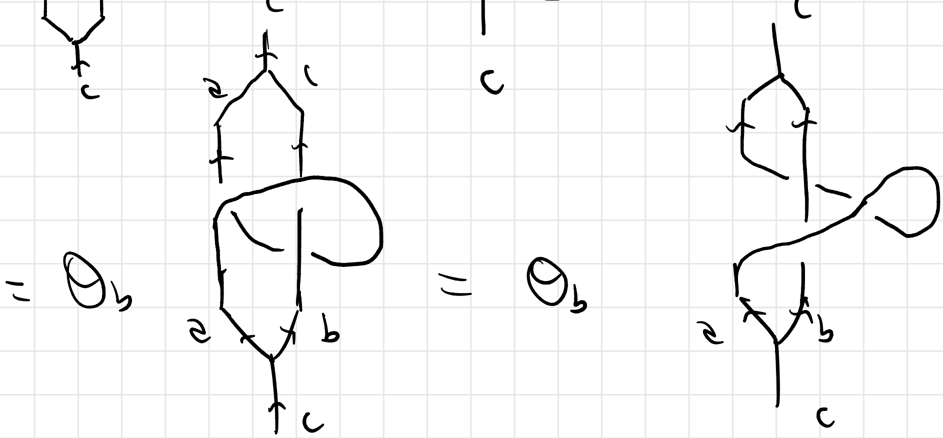
⇒ One can show that

$$R^{aa^*}_1 = \theta_a^* \chi_a \quad \swarrow \text{Specified by } \boxed{\operatorname{sgn}\left(F^{aaa}_1\right)_{11}}$$

↳ ±1 Frobenius Schur indicator of $a \in \mathcal{L}$.

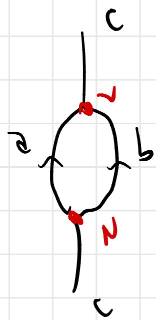
they obey Ribbon equation.





$$= \theta_b\, \theta_a$$ 

$$= \theta_a\, \theta_b\, R^{ab}_c$$ 

$$= \theta_a\, \theta_b\, R^{ab}_c\, R^{ba}_c$$ 

$$\Rightarrow \boxed{R^{ab}_c \cdot R^{ba}_c = \frac{\theta_c}{\theta_a \theta_b}\, \mathbb{1}}$$

identity in $\mathrm{Hom}(a \otimes b, c)$

Ribbon equation.

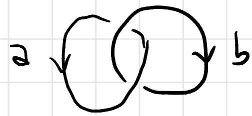
We define $|L| \times |L|$

$$T_{ab} = \theta_a\, \delta_{ab}$$

a, b assume values in L.

Similarly, we diagramatically define $|\mathcal{C}| \times |\mathcal{C}|$ matrix

$$S_{ab} = \frac{1}{D}$$



$$D = \sqrt{\sum_a d_a^2}$$

One can show (Homework)

$$S_{ab} = \frac{1}{D} \sum_c N^{ab}_c \, \mathrm{Tr}\left(R^{ab}_c \, R^{ba}_c \right) d_c$$

↑ matrix indices

$\mu, \nu = 1, \dots, N^{ab}_c$

$$S_{ab} = \frac{1}{D} \sum_c N^{ab}_c \, \frac{\theta_c}{\theta_a \theta_b} \, d_c$$

These S and T matrices must obey

$S^2 = C$ ~ charge-conjugation matrix.

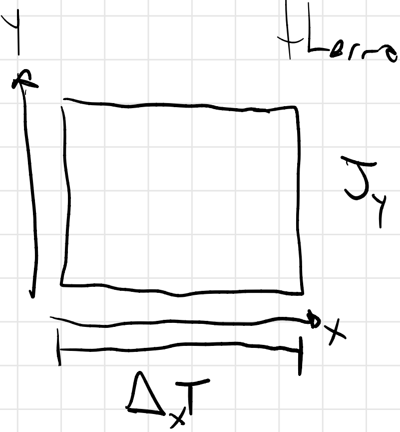
$$(ST)^3 \stackrel{!}{=} e^{\frac{2\pi i c_-}{8}} C$$

S & T matrices projective representations of modular group $SL(2, \mathbb{Z})$

c_- : chiral central charge

- $c_- \neq 0 \Rightarrow$ chiral CFT at boundary.
- $c_- \neq 0 \Rightarrow$ $\underbrace{\text{very robust}}_{\text{arbitrary perturbations}}$ edge modes

  [Levin 1301.7355]
- Experimentally $c_- \neq 0$ implies in a thermal Hall conductance κ_{xy}



$$J_y = \kappa_{xy} \, \Delta_x T$$

$$\kappa_{xy} = \frac{c_- \pi^2 k_B^2 T}{3h}.$$

- c_- is only defined mod 8

$$c_- \to c_- + 8$$

$\hookrightarrow$ $\exists$ topological phase E_8 with no non-trivial anyons in the bulk, but non-trivial boundary w/ $c_- = 8$

$\hookrightarrow$ E_8 ~ "Hidden topological phase"

$\Rightarrow$ E_8 level 1 WZW model

$\Rightarrow$ K-matrix CS theory.

$$K = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \leftarrow \det K = 1$$

$$\operatorname{sgn} K = 8$$

$\operatorname{sgn} K$ = # positive eigenvalues − # negative eigenvalues

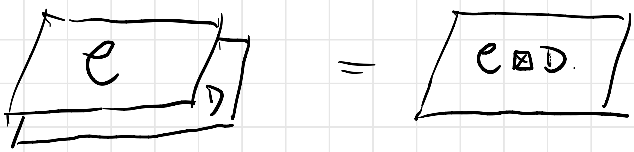
$\hookrightarrow$ 16-fold way classification (invertible bosonic phases)

$\hookrightarrow \mathcal{e}$ on $L = \{1\}$

$\Rightarrow$ <u>Stacking</u>

Consider the set S of all topological orders.

We can equip this set with $\boxtimes$ (Deligne product) which to a stacking operation



$$\mathcal{e} = \left(L,\ N^{ab}_c,\ F^{abc}_d,\ R^{ab}_c\right)$$

$$D = \left(L',\ N^{ab'}_c,\ F^{abc'}_d,\ R^{ab'}_c\right)$$

$$\to \underbrace{\mathcal{e} \boxtimes D}_{\text{"new topological order"}} = \left(L \times L',\ N^{ab}_c N^{ab'}_c,\ F^{abc}_d \otimes F^{abc'}_d,\ R^{ab}_c \otimes R^{ab'}_c\right)$$

$$G \overset{\text{local unitary}}{\simeq} \mathcal{e} \boxtimes D.$$

$\Rightarrow \mathcal{C}_{triv} = (\{1\}, N_{11}^{1} = 1, F \sim 1, R \sim 1)$
behaves as unit element.

$\Rightarrow$ S with $\boxtimes$ $\rightsquigarrow$ Monoid. (group with elements not necessarily having inverses).

Eg: $\mathcal{C}$ = Toric, $\mathcal{D}$ = Fib.
4, 2

$\Rightarrow$ $\mathcal{C} \boxtimes \mathcal{D}$
8 anyons

there exists a subset A of S whose elements all have inverses. A forms an Abelian group

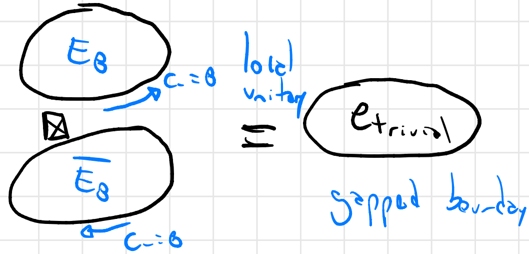
top. order $\in A$ $\rightsquigarrow$ Invertible topological orders.
$\hookrightarrow$ for all $\mathcal{C} \in A$, there exists an inverse $\bar{\mathcal{C}}$ such that

local unitary

$$\mathcal{C} \boxtimes \bar{\mathcal{C}} \cong \mathcal{C}_{trivial}$$

Necessary condition for $\mathcal{C} \in A \Rightarrow |\mathcal{C}| = 1.$

E.g.: • E_8 state



• IQHE

• Topological superconductors.

$\Rightarrow$ Invertible phases; non-trivial edge modes with no protecting symmetries; trivial bulk (no anyons).

Ex: $\mathcal{C} = \overset{c\ 7}{\mathrm{Fib}}$, $\mathcal{D} = \overset{c\ 5}{\mathrm{Ising}}$,

$\mathcal{C} \boxtimes \mathcal{D}$ has 6 anyons $\underset{\in \mathrm{Fib}}{a} \boxtimes \underset{\in \mathrm{Ising}}{b}$

Eg:

$\mathcal{C} \boxtimes \mathcal{D} \ni \tau \boxtimes \sigma$ Consider fusion:

$$(\tau \boxtimes \sigma) \otimes (\tau \boxtimes \sigma) = (\tau \otimes \tau) \boxtimes (\sigma \otimes \sigma)$$

$$= (1+\tau) \boxtimes (1+\psi)$$

$$= 1 \boxtimes 1 + 1 \boxtimes \psi + \tau \boxtimes 1 + \tau \boxtimes \psi$$