

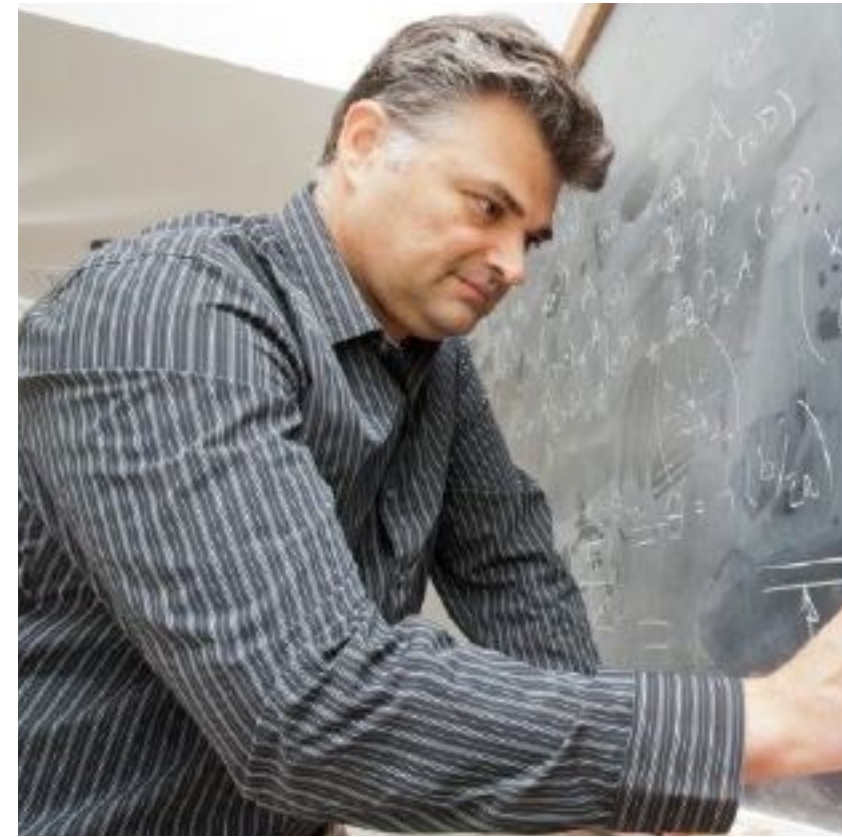
Gauging Modulated Symmetries

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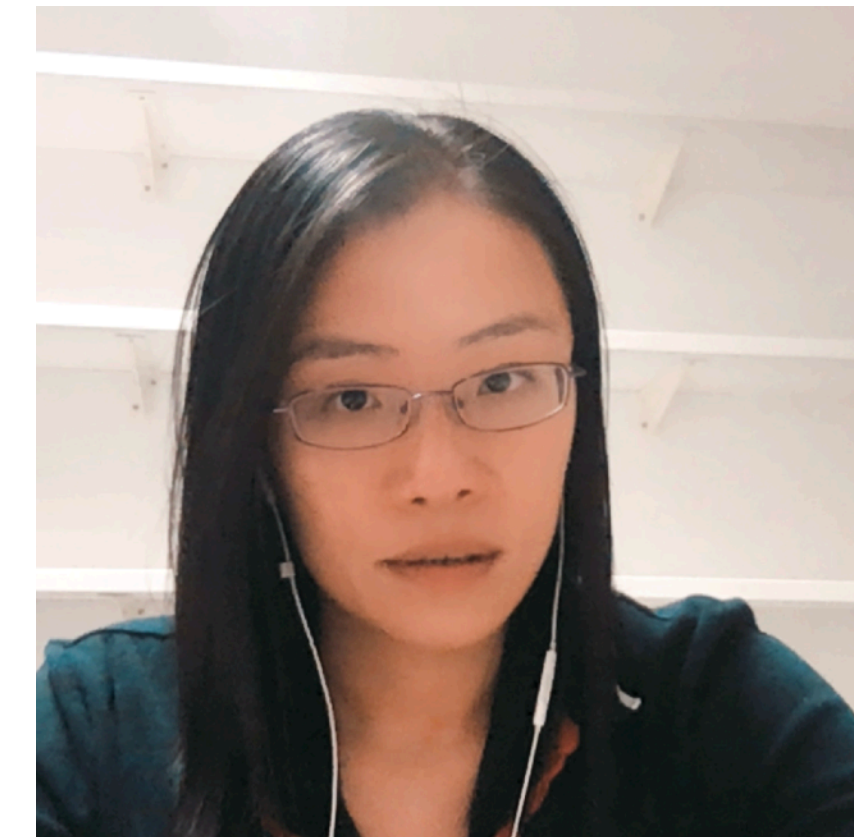
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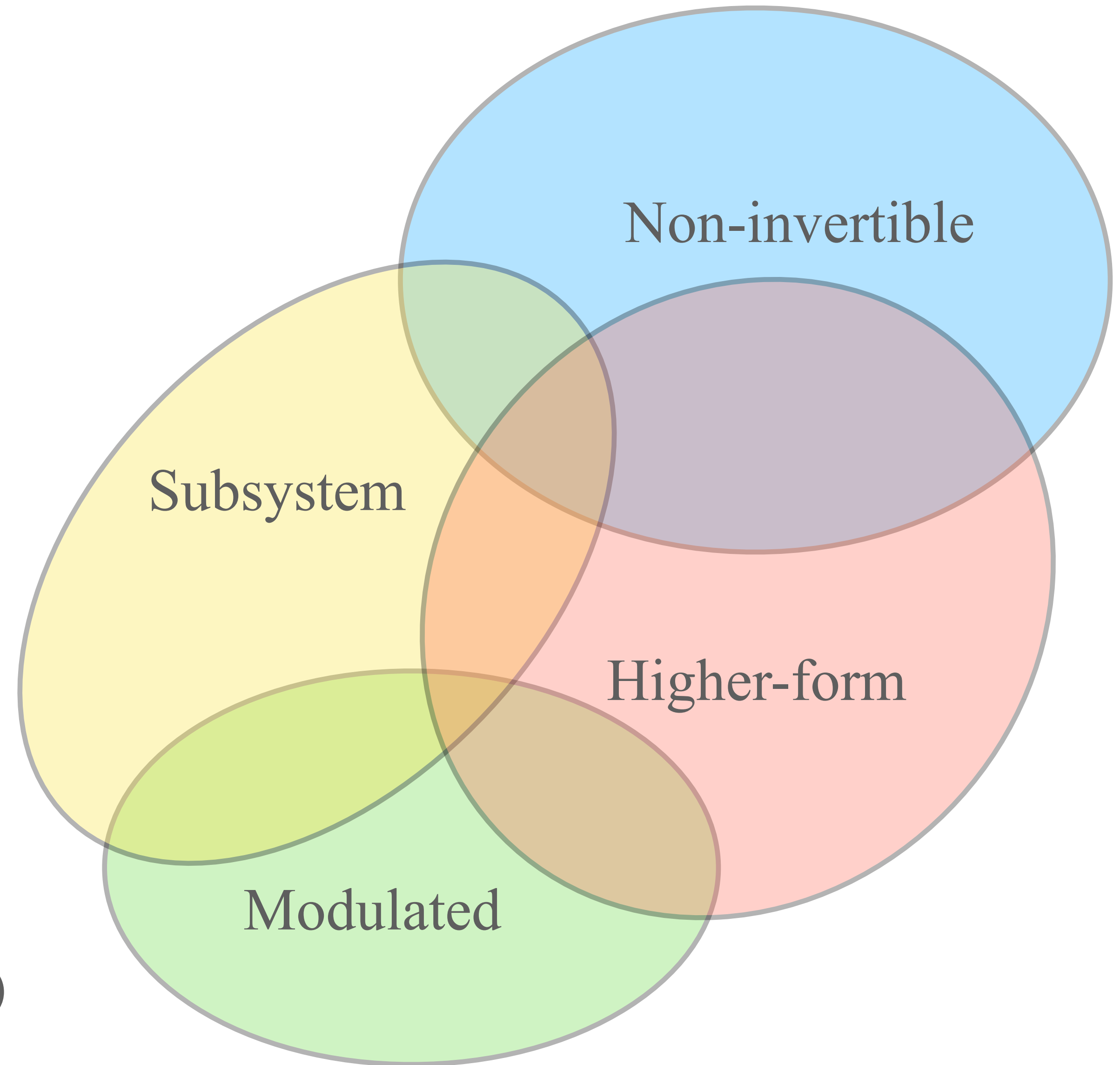
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Introduction

- Generalized symmetries extend the conventional notion of symmetry in several distinct ways:
 - Fusion ring multiplication rules (**non-invertible**)
 - Support on topological sub-manifolds (**higher-form**)
 - Support on rigid subspaces (**subsystem**)
 - Non-uniform in space (**modulated**)



Introduction

- Typically, internal and space symmetry groups G_{int} and G_{space} do not talk to each other
- Spatially **modulated symmetries**, on the other hand, act non-uniformly in space and have a non-trivial interplay with space-like symmetries
- This is incorporated through a **group homomorphism** which defines a non-trivial action $\varphi : G_{\text{space}} \rightarrow \text{Aut}(G_{\text{int}})$
- Total symmetry group $G = G_{\text{int}} \rtimes_{\varphi} G_{\text{space}}$
- In general G_{int} can be either finite or infinite, Abelian or non-Abelian, 0-form or higher form.

Motivation

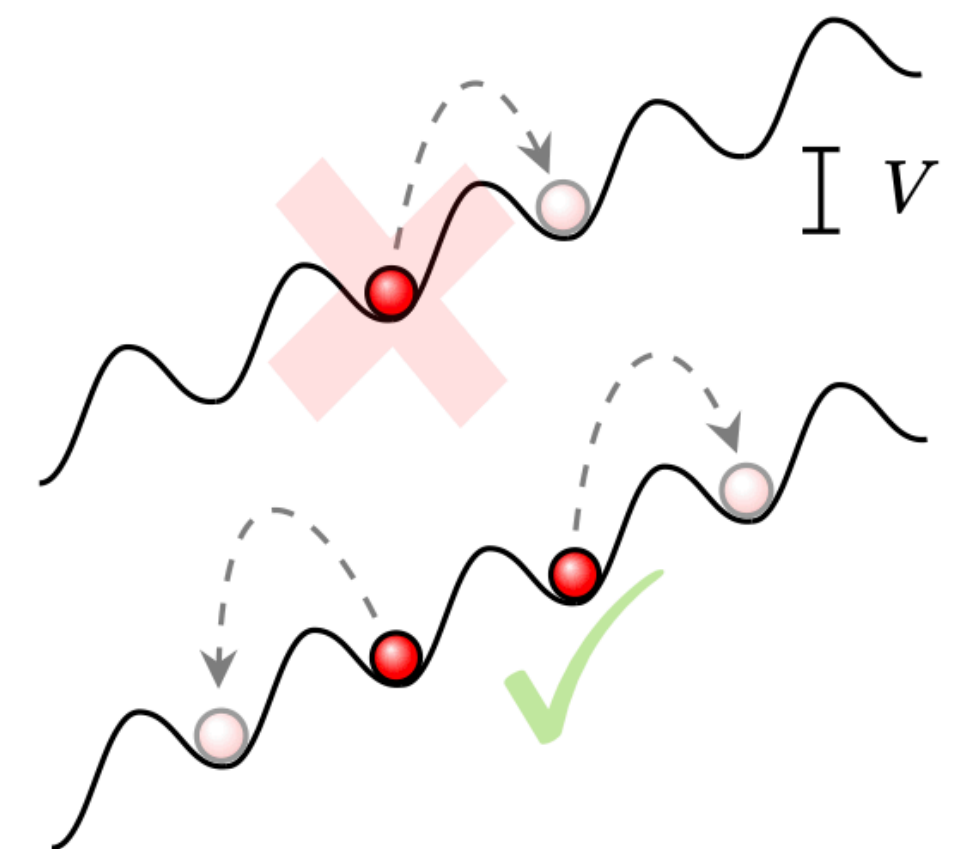
Spatially modulated symmetries play an important role in systems with **multicritical points**

$\sim z \neq 1$ dynamical critical exponents (Lifshitz, 1941)

- **constrained dynamics:**

\sim Fractons (Pretko, 2017); Hilbert space fragmentation (Khemani, Hermele & Nandkishore, 2020); tilted lattices (Lake, Lee, Han & Senthil, 2023)

- This talk: **Route for crystalline SETs** (Symmetry Enriched Topological Order enriched by crystalline symmetries G_{space})



Dipole Moment

Consider $G_{\text{int}} = U(1) \times U(1)^d$ generated by **dipole moment** and **charge** in $(d+1)$ dimensions

$$U_q = \exp(i \alpha_0 Q), \quad \text{and} \quad U_{\text{dip}}^{(i)} = \exp(i \alpha_i D_i) \quad \text{with} \quad Q = \int d\mathbf{x} \rho(\mathbf{x}) \quad \text{and} \quad D_i = \int d\mathbf{x} x_i \rho(\mathbf{x})$$

- Translations $T_{\mathbf{a}} : \mathbf{x} \rightarrow \mathbf{x} + \mathbf{a}$ have a non-trivial action $T_{\mathbf{a}} U_{\text{dip}}^{(i)} T_{\mathbf{a}}^\dagger = U_{\text{dip}}^{(i)} U_q^{\mathbf{a}_i}$

Total symmetry group $G = (U(1) \times U(1)^d) \rtimes_{\varphi} G_{\text{space}}$

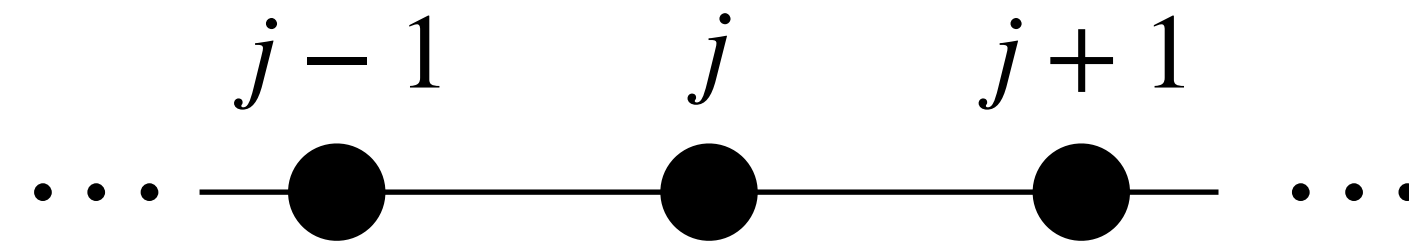
\Rightarrow More generally, the modulation can be implemented by **arbitrary functions**

$$G^{(q)} = \int d\mathbf{x} f^{(q)}(\mathbf{x}) \rho(\mathbf{x}), \quad q = 1, \dots, n, \quad \text{with} \quad f^{(q)} : \text{space} \rightarrow \text{target space}$$

\sim From now on, we focus on lattice systems with G_{int} Abelian and finite

Warm-up: 1D chain

- To warm-up, let us consider one-dimensional lattice chains Λ



- On each site there is a \mathbb{Z}_N **degree of freedom** (*qudit*) $\mathcal{H}_j \cong \mathbb{C}^N$

$$Z_j X_k = e^{\frac{2\pi i}{N} \delta_{jk}} X_k Z_j \quad \text{and} \quad X_j^N = Z_j^N = 1$$

- We consider **finite Abelian groups**, which constrains the functions in symmetry operators

$$U_f = \prod_j X_j^{f_j} \quad \text{to be} \quad f: \Lambda \rightarrow \mathbb{Z}/N\mathbb{Z}$$

Spatially Modulated Symmetries

The symmetry $U_f = \prod_j X_j^{f_j}$ acts on local operators as

$$Z_j \mapsto e^{\frac{2\pi i}{N} f_j} Z_j \quad \text{and} \quad X_j \mapsto X_j$$

Takeaway: The symmetry acts differently in different points in space

- We want to preserve crystalline symmetries \Rightarrow one must conserve not only U_f but also $U_{g(f)}$ where $g(f)$ is the function f transformed by crystalline action $g \in G_{\text{space}}$
- Thus, the set of operator symmetries $U_q = \prod_j X_j^{f_j^{(q)}}$ must **close under** G_{space} . We denote the set of modulating functions $S = \{f^{(q)}\}_{q=1,\dots,r}$.

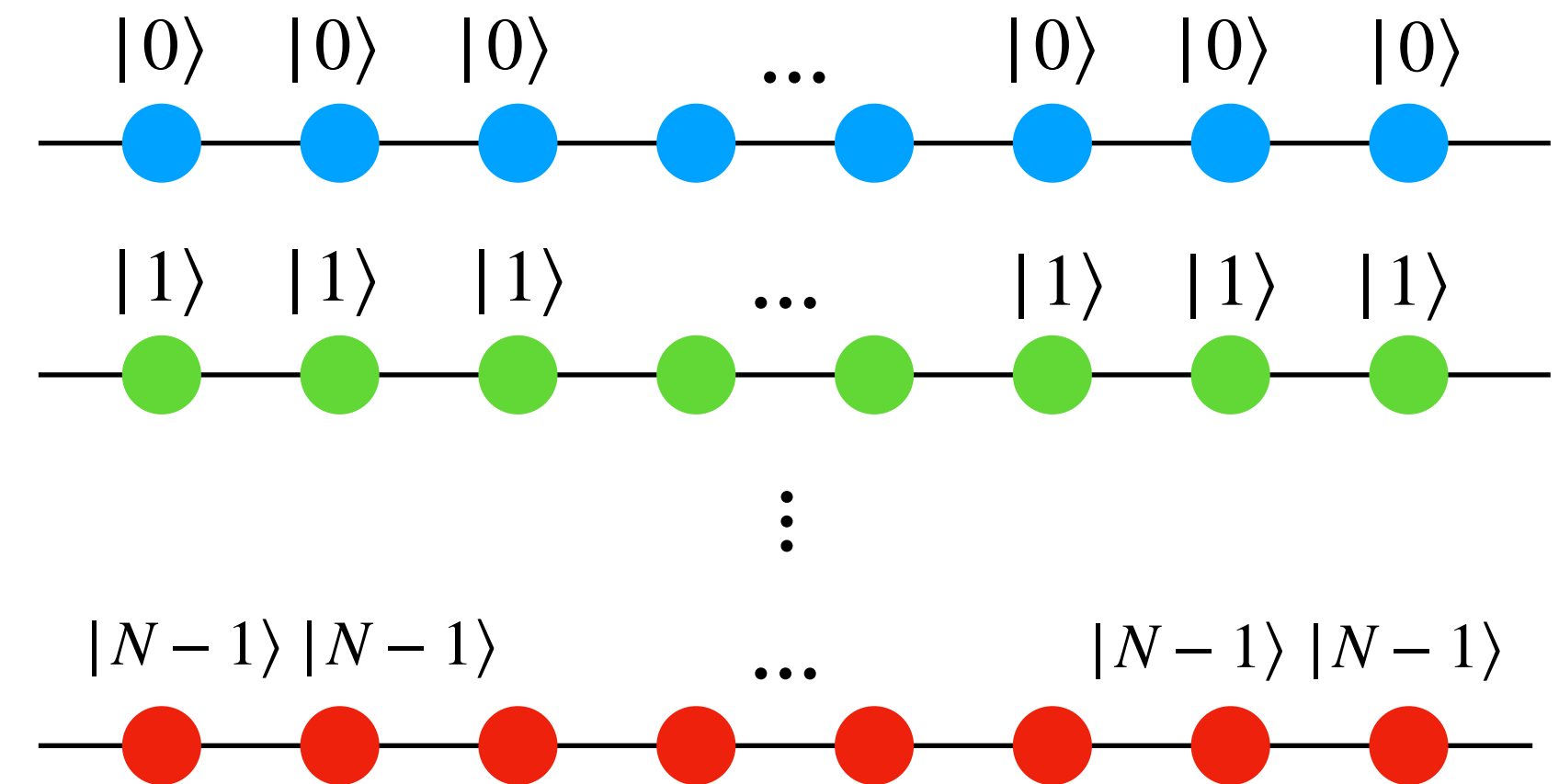
\mathbb{Z}_N Ising in 1+1D

- Prototypical example: Hamiltonian that is invariant under U for a uniform $f_j = 1$

$$H(g) = - \sum_j Z_{j-1}^\dagger Z_j - g \sum_j X_j + \text{h.c.}$$

From $U = \prod_j X_j$, it follows that $UHU^\dagger = H$

Internal group $G_{int} = \mathbb{Z}_N$ with trivial action under translations $TUT^\dagger = U$. Thus $G = \mathbb{Z}_N \times \mathbb{Z}$

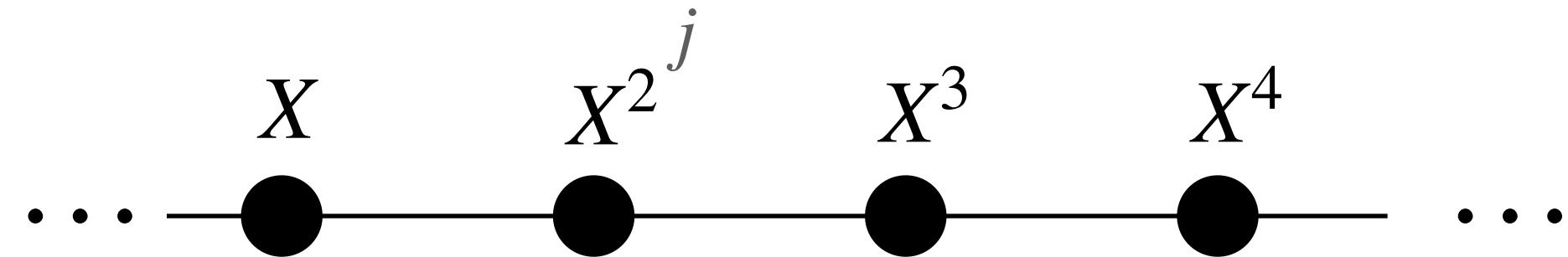


- In 1D, in the $g \ll 1$ limit, SSB and N-fold degenerate ground states with $\langle Z_i^\dagger Z_j \rangle \sim 1$

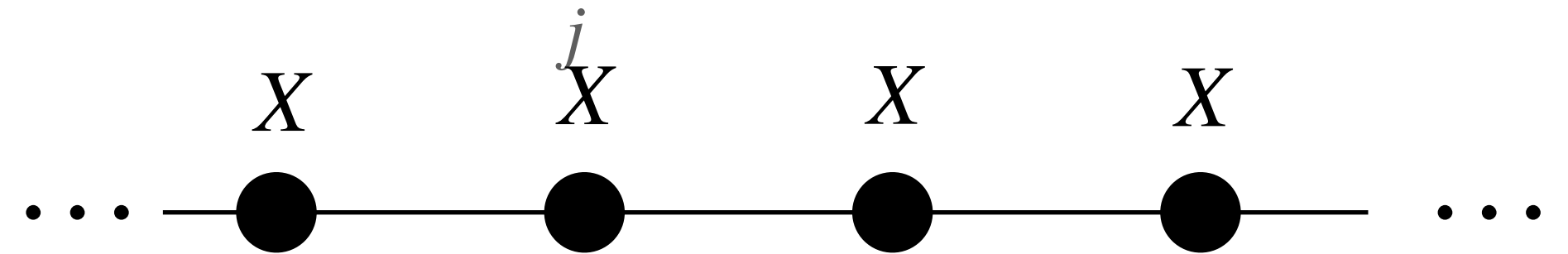
\mathbb{Z}_N Dipole in 1+1D

- Consider now “ \mathbb{Z}_N dipole moment” $f_j = j$

$$D = \prod X_j^j \quad (\text{“dipole”})$$



$$Q = \prod X_j \quad (\text{“charge”})$$



- $D^N = Q^N = I \quad \Rightarrow \quad G_{int} = \mathbb{Z}_N \times \mathbb{Z}_N$
- Both these operators commute with $H(g)$ and obey

$$H(g) = - \sum_j Z_{j-1} Z_j^\dagger{}^2 Z_{j+1} - g \sum_j X_j + \text{h.c.}$$

Structure of Modulation

- **Translations** $T : j \rightarrow j + 1$ implies an homomorphism which encodes

$$T D T^\dagger = D Q$$

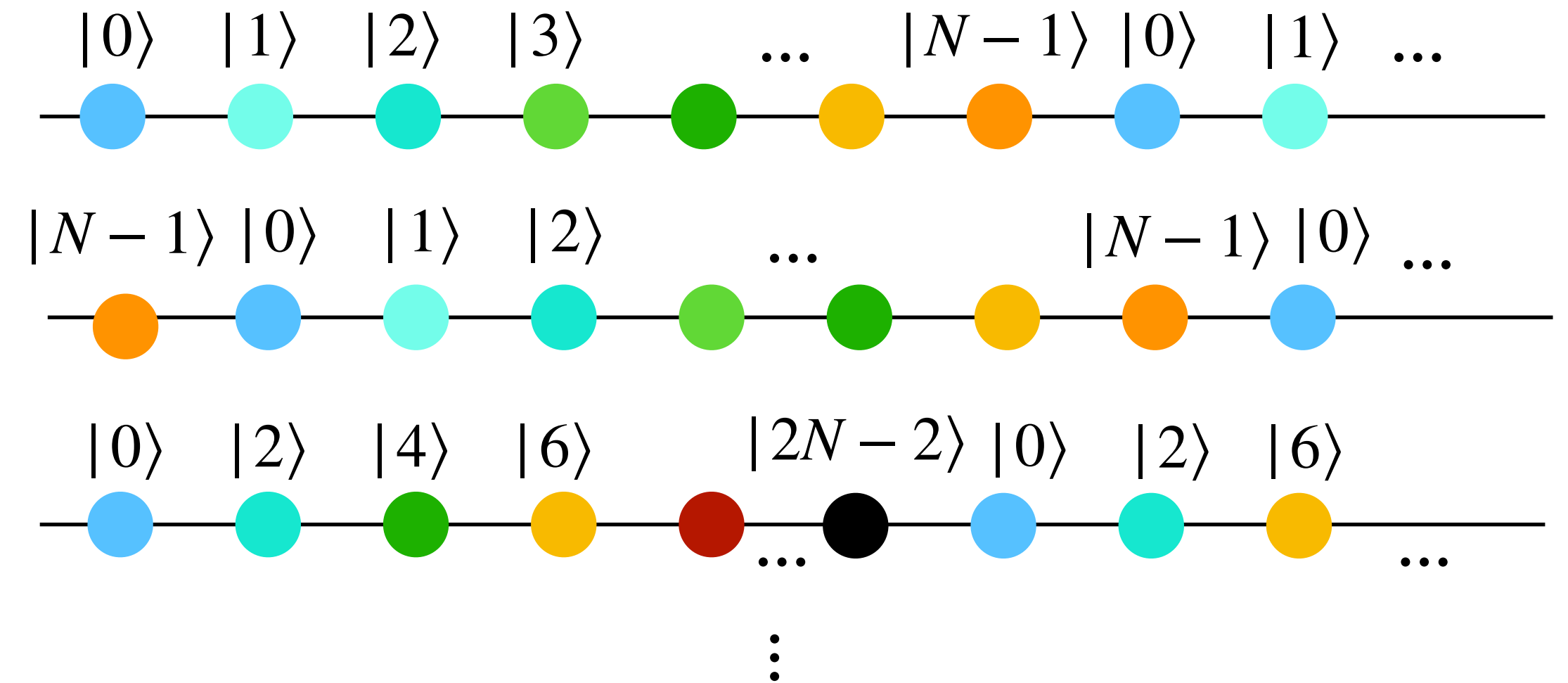
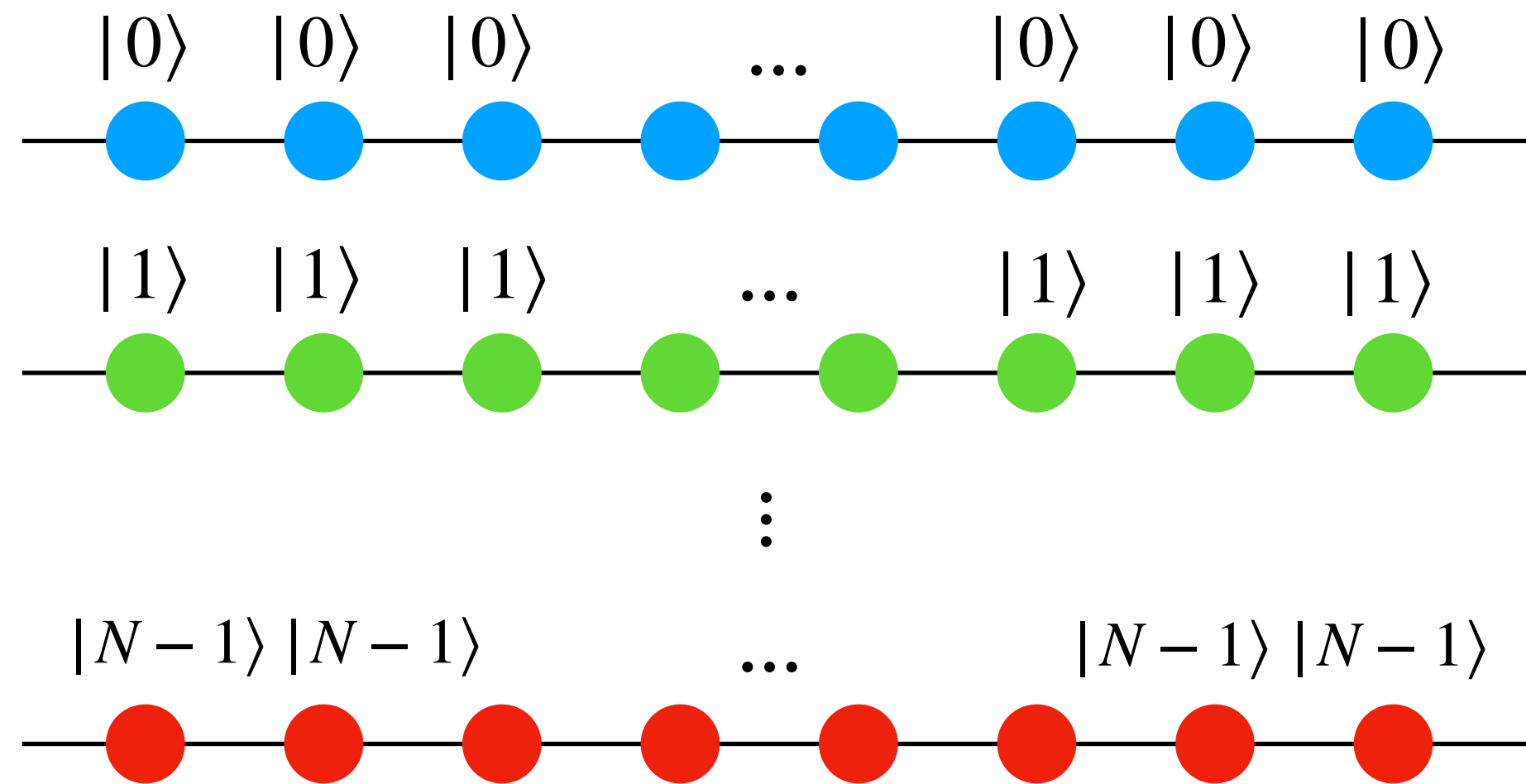
$\sim \varphi(T)$ is an **automorphism** in $\mathbb{Z}_N \times \mathbb{Z}_N$ that maps the **symmetry generators**

$$\varphi(T) : Q \mapsto Q \quad \text{and} \quad \varphi(T) : D \mapsto QD$$

\Rightarrow Total symmetry group $G = (\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes_{\varphi} \mathbb{Z}$

Dipole Symmetry SSB

- SSB of G_{int} ($g \ll 1$ limit):



There are N^2 degenerate ground states, which transform among themselves under translations

\Rightarrow Spatially modulated degenerate ground states (Lake, Hermele & Senthil, 2022)

Examples in (2+1)D

Let $\mathbf{x} = (x, y) \in \mathbb{Z} \times \mathbb{Z}$ label the coordinates of a two-dimensional lattice. For concreteness, consider square lattice $G_{\text{space}} = C_4 \rtimes \mathbb{Z}^2$

- Examples of **modulating functions** (which transform according to Irreps of point-groups):

$f_{\mathbf{x}} = 1,$	Charge
$f_{\mathbf{x}} = x, y \text{ mod } N$	Dipole
$f_{\mathbf{x}} = x^2 - y^2, xy \text{ mod } N$	Quadrupole
\vdots	
$f_{\mathbf{x}} = \text{degree } n \text{ poly mod } N$	2^n -Multipole
\vdots	
$f_{\mathbf{x}} = a^{x+y} \text{ mod } N, \quad a \in \mathbb{Z}$	Exponentially modulated charge

Symmetric Hamiltonians

Let $S = \{f^{(q)}\}_{q=1,\dots,r}$ be the set of functions such that $U_q = \prod_j X_j^{f_j^{(q)}}$ are symmetries. Symmetric Hamiltonians:

$$H_{\text{matter}} = -t \sum_{\mathbf{x}, a} \prod_i Z_{\mathbf{x}+i\hat{a}}^{\Delta_a^i} - g \sum_{\mathbf{x}} X_{\mathbf{x}} + \text{h.c.} + \dots$$

where the Δ_a^i are \mathbb{Z}_N coefficients that define a “**lattice derivative**” operator $\sum_i \Delta_a^i f_{\mathbf{x}+i\hat{a}} \equiv \Delta_a f$

- The coefficients Δ_a^i are determined by the condition $\Delta_a f^{(q)} = 0$, for $1 = 2, \dots, r$.
- Δ_a defines $\mathcal{O}(r)$ -body interactions \Rightarrow non-periodic functions require non-local Hamiltonians!

Symmetry	$f_{\mathbf{x}}$	$\Delta_a f$	$\prod_i \hat{Z}_{\mathbf{x}+i\hat{a}}^{\Delta_a^i}$
Charge	1	$f_{\mathbf{x}+\hat{a}} - f_{\mathbf{x}}$	$\hat{Z}_{\mathbf{x}+\hat{a}} \hat{Z}_{\mathbf{x}}^\dagger$
Dipole	x, y	$f_{\mathbf{x}+\hat{a}} - 2f_{\mathbf{x}} + f_{\mathbf{x}-\hat{a}}$	$\hat{Z}_{\mathbf{x}+\hat{a}} \hat{Z}_{\mathbf{x}}^{\dagger 2} \hat{Z}_{\mathbf{x}-\hat{a}}$
\vdots			
Exponential	m^{x+y} , for $m \in \mathbb{N}$	$f_{\mathbf{x}+\hat{a}} - m f_{\mathbf{x}}$	$\hat{Z}_{\mathbf{x}+\hat{a}} \hat{Z}_{\mathbf{x}}^{\dagger m}$

(GD, Chamon & You, 2023)

SETs: Beyond SSB

Gauging G_{int} in modulated symmetries in 2+1D is a mechanism to get topologically ordered with locally constrained anyons

Deconfined phase of gauge theory \leftrightarrow Topological Order

The resulting topological orders are enriched by G_{space} and correspond to **crystalline symmetries enriched topological order** (SETs)

Under lattice symmetry actions $g \in G_{\text{space}}$, anyons permute flavors

$$g : a \mapsto {}^g a$$

so that topological data is preserved.

Gauging G_{int}

- Gauging G_{int} corresponds to a two step process:
 1. I) **Extend Hilbert space** by introducing \mathbb{Z}_N gauge fields A_a and E_a
 2. II) **Impose Gauss law**, generator of gauge transformations

$$Z_{\mathbf{x}} \rightarrow e^{i\alpha_{\mathbf{x}}} Z_{\mathbf{x}} \quad \text{and} \quad A_{a,\mathbf{x}} \rightarrow A_{a,\mathbf{x}} + \Delta_a \alpha_{\mathbf{x}}$$

- Minimally coupled theory

$$H_{\text{matter+gauge}} = - \sum_{\mathbf{x},a} e^{-iA_{a,\mathbf{x}}} \prod_i \hat{Z}_{\mathbf{x}+i\hat{a}}^{\Delta_a^i} + \text{h.c.} + \dots$$

(2+1)D Modulated Gauge Theories

- From the canonical relation $[A_{a,\mathbf{x}}, E_{b,\mathbf{y}}] = i\delta_{a,b}\delta_{\mathbf{x},\mathbf{y}}$, gauge transformations define a **Gauss law**

$$q_{\mathbf{x}} = \Delta_a E_a.$$

- One can also define a gauge invariant **magnetic flux**, schematically expressed as

$$b_{\mathbf{x}} = \check{\Delta}_a A_a$$

where $\check{\Delta}_a$ is defined such that $\check{\Delta}_a \Delta_a \alpha = 0$ for any lattice function $\alpha_{\mathbf{x}}$, which enforces that $b_{\mathbf{x}}$ is gauge invariant.

- Δ_a and $\check{\Delta}_a$ depend on the functions $f_{\mathbf{x}}$ and the lattice symmetries one wishes to preserve

Deconfined phase

- These define \mathbb{Z}_N star and plaquette operators with $X_a \sim e^{2\pi i E_a/N}$, and $Z_a \sim e^{iA_a}$

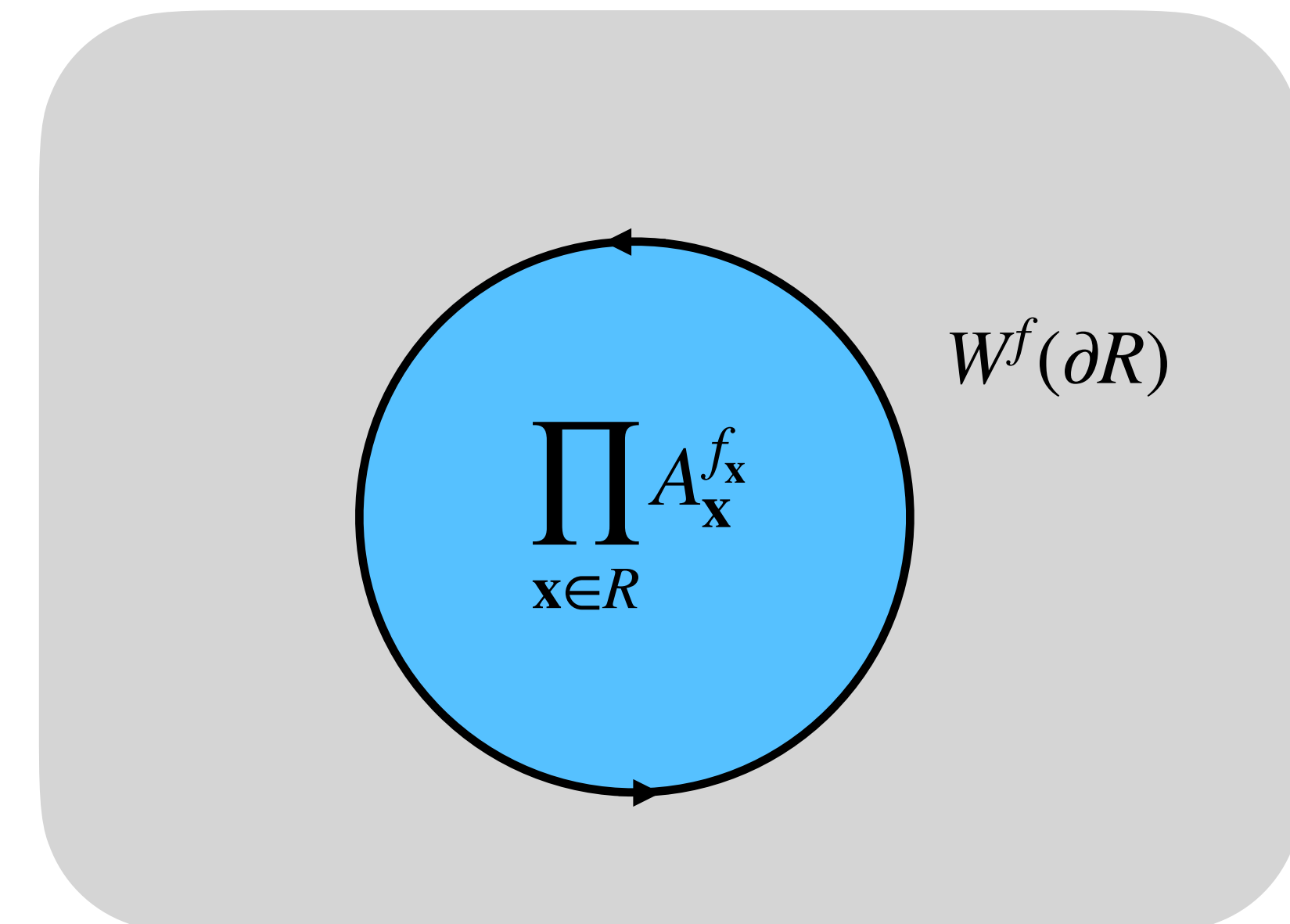
$$A_{\mathbf{x}} = e^{\frac{2\pi i q_{\mathbf{x}}}{N}} = \prod_a X_a^{\Delta_a} \quad \text{and} \quad B_{\mathbf{x}} = e^{i b_{\mathbf{x}}} = \prod_a Z_a^{\check{\Delta}_a}$$

- And let $\tilde{f} \in \text{Ker}(\check{\Delta})$, under periodic boundary conditions $\prod_{\mathbf{x} \in \Lambda} A_{\mathbf{x}}^{\tilde{f}_{\mathbf{x}}} = 1$, and $\prod_{\mathbf{x} \in \Lambda} B_{\mathbf{x}}^{\tilde{f}_{\mathbf{x}}} = 1$.

Finally, for ∂R be the boundary of a region R subset of Λ , then

$$\prod_{\mathbf{x} \in R} A_{\mathbf{x}}^{\tilde{f}_{\mathbf{x}}} = W^{\tilde{f}}(\partial R) \quad \text{and} \quad \prod_{\mathbf{x} \in R} B_{\mathbf{x}}^{\tilde{f}_{\mathbf{x}}} = V^{\tilde{f}}(\partial R),$$

define **1-form symmetries** $W^{\tilde{f}}$ and $V^{\tilde{f}}$ of the gauge theory



Modulated 1-form symmetries

- The higher-form symmetries W^f and $V^{\tilde{f}}$ depend explicitly on the functions f and \tilde{f} and are **modulated**

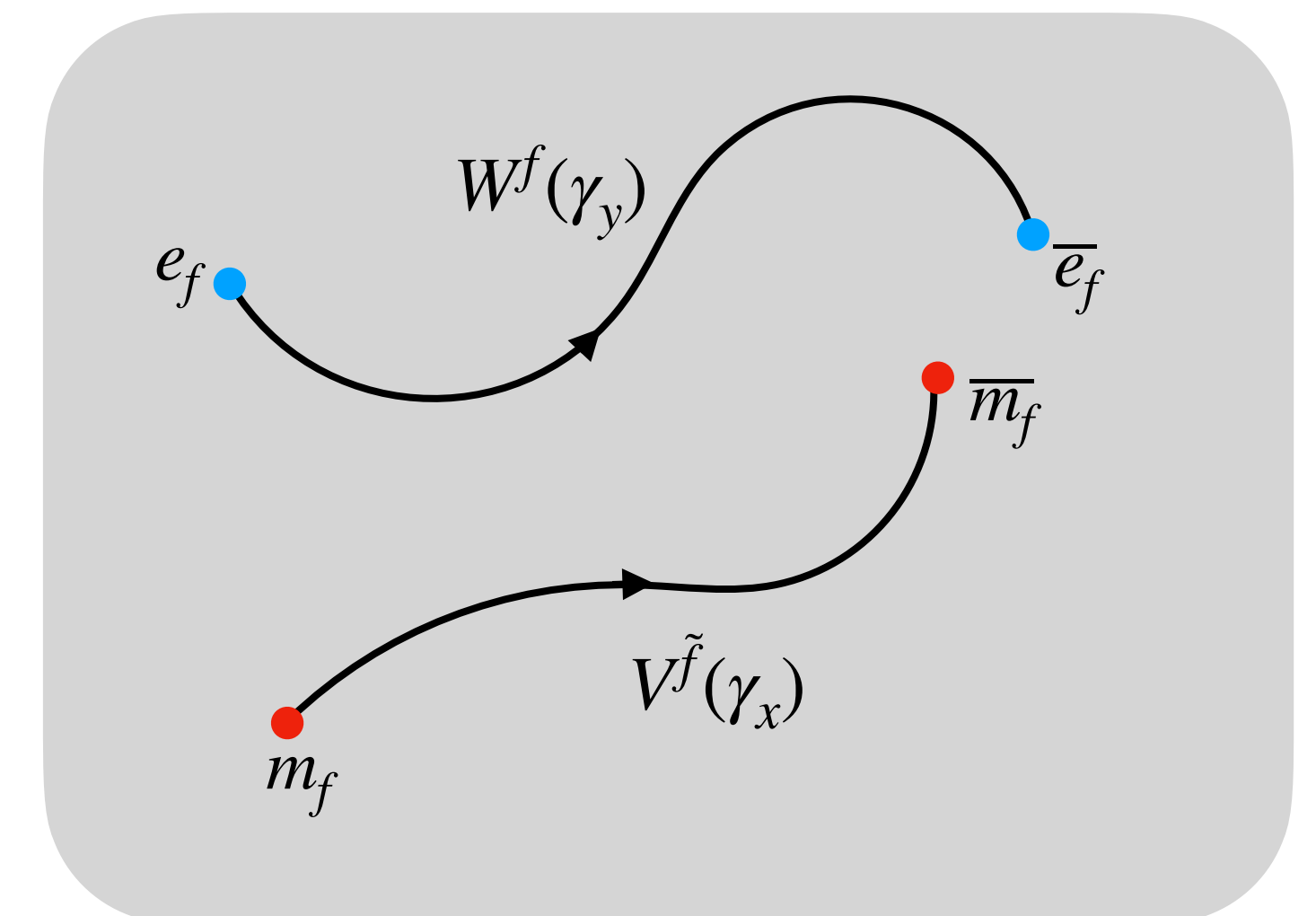
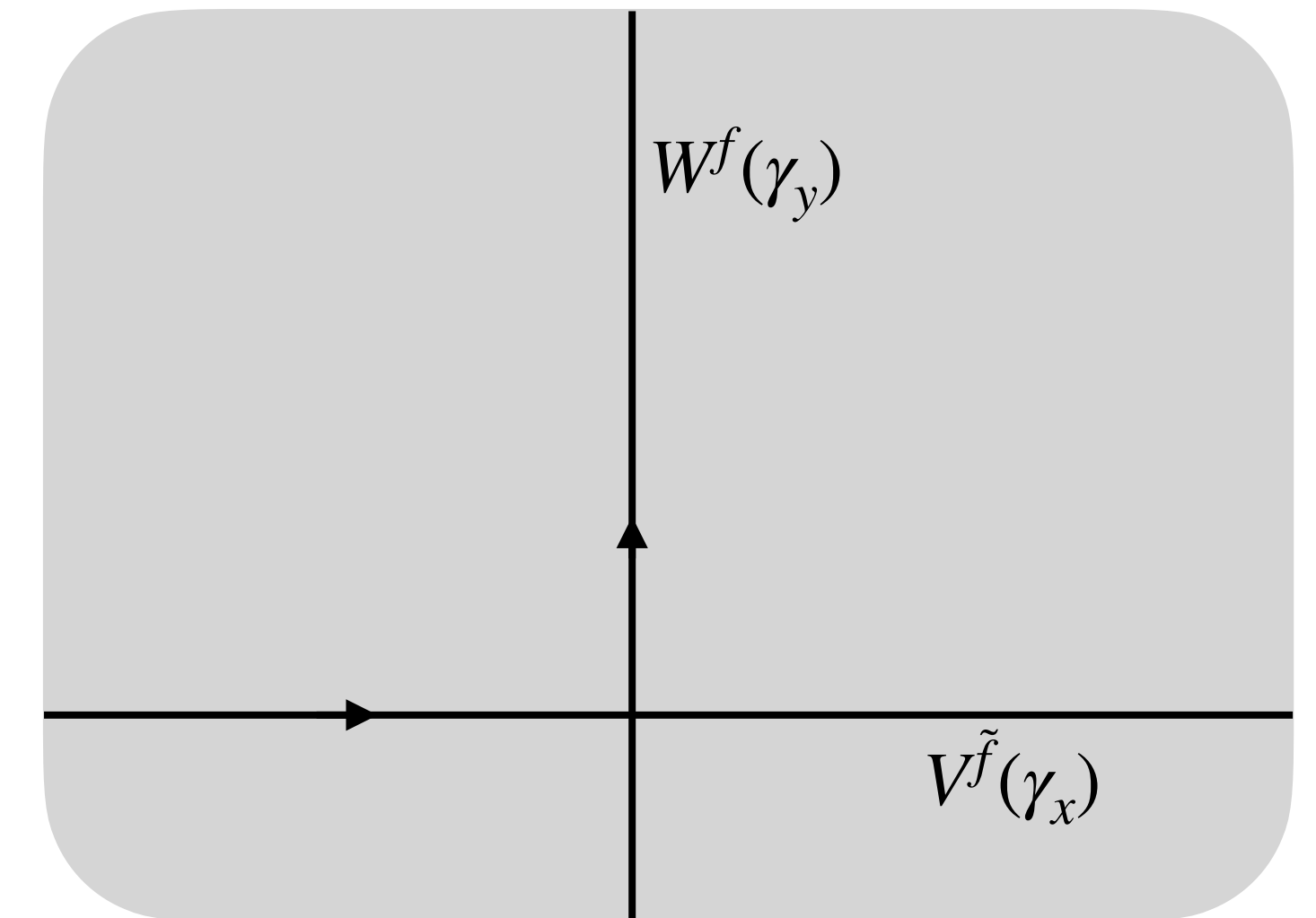
$$TW^fT^\dagger = W^{T(f)}$$

- Additionally, they share a **mixed anomaly**

$$W^f(\gamma)V^{\tilde{f}}(\gamma^*) = \omega^{I(\gamma,\gamma^*)}V^{\tilde{f}}(\gamma^*)W^f(\gamma),$$

where $I(\gamma, \gamma^*)$ counts the lattice intersection and $\omega = e^{2\pi i/N}$.

- Open strings $W^f(\gamma)$ and $V^{\tilde{f}}(\gamma^*)$ create **Abelian anyonic excitations** at the endpoints of γ and γ^*



(1+1)D Gauging

In 1+1D the story is a bit simpler (translations only):

$$G = G_{\text{int}} \rtimes_{\varphi} G_{\text{space}} \xrightarrow{\text{gauge } G_{\text{int}}} G^{\vee} = G_{\text{int}} \rtimes_{\varphi^{\vee}} G_{\text{space}}$$

- In general, the symmetry can have different modulation φ and φ^{\vee}
 - ~ For N prime φ and φ^{\vee} are isomorphic, both theories are **self-dual** under gauging (KW-like duality);
 - ~ We also explicitly find **KW non-invertible symmetries**, as well as their dependence on the modulation structure;

(2+1)D Gauging

- This procedure provide us exactly solvable lattice models for crystalline SETs

$$G = G_{\text{int}} \rtimes_{\varphi} G_{\text{space}} \xrightarrow{\text{gauge } G_{\text{int}}} G^{\vee} = \left(G_{\text{int}}^{(1)} \times G_{\text{int}}^{(1)} \right) \rtimes_{\varphi^{\vee}} G_{\text{space}}$$

Examples:

$$\mathbb{Z}_N \text{ Rank 2 Toric Code} \leftrightarrow G_{\text{int}} = \mathbb{Z}_N^3 \text{ with } f_{\mathbf{x}} = 1, x, y$$

(Oh, Kim, Moon & Han, 2022)

$$\mathbb{Z}_N \text{ dipolar-quadrupolar code} \leftrightarrow G_{\text{int}} = \mathbb{Z}_N^4 \text{ with } f_{\mathbf{x}} = 1, x, y, xy$$

(GD, Fontana, Gomes & Chamon, 2023)

Crystalline SETs

- Anyons have **restricted mobility**, incorporated in the Gauss law and magnetic flux through Δ and $\check{\Delta}$
- Ground state degeneracy depends on system size (flavor of **UV/IR mixing**)
- Anyons have **position-dependent quantum numbers** (Pace & Wen, 2022)
- Low energy effective theories are equivalent to **twisted TQFT** gauge theories
- Different crystalline SETs can be connected through phase transitions driven by **anyon condensation** (GD & You, 2024)

Subsystem \sim Modulated

- Fracton topological order in (3+1)D is understood in terms of **subsystem symmetries** (Vijay, Haah & Fu, 2016)

- Consider a subsystem symmetry operator $U_\Omega = \prod_{\mathbf{x} \in \Omega} X_{\mathbf{x}}$ with support on a sub-lattice

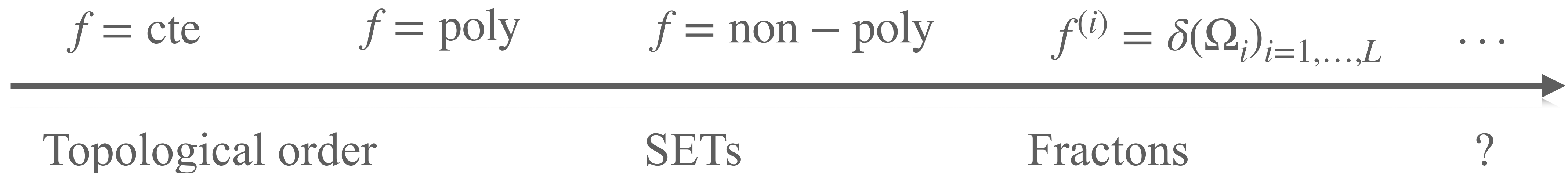
$\Omega \subset \Lambda$ of the whole lattice Λ

$$\text{(subsystem)} \quad \prod_{\mathbf{x} \in \Omega} X_{\mathbf{x}} = \prod_{\mathbf{x} \in \Lambda} X_{\mathbf{x}}^{\delta_{\mathbf{x}}(\Omega)} \quad \text{(modulated)}$$

\Rightarrow 3D fracton systems can be casted as a modulated gauge theories associated to infinitely many modulated symmetries $f_{\mathbf{x}}^{(i)} = \delta_{\mathbf{x}}(\Omega_i)$ for an extensive number of sub-lattices $\Omega_i \subset \Lambda$, $i = 1, \dots, L$.

Final Considerations

- Gauging modulated symmetries is cheap way of constructing exactly solvable models for **crystalline SETs**
- Classification and characterization of such phases are mapped into the study of **functions and number theory**;
- Modulated gauge theories can take into account a **whole spectrum of enriched Abelian topological phases**



Future Perspectives

- Flow into **continuum**: Does modulated symmetries have an exact realization in the continuum?
 - ⇒ May offer insights into type-II fracton field theories, with fractal subsystem symmetries
- Investigate the physics of **critical points**: non-CFTs continuous phase transitions?
 - ⇒ While polynomial symmetries are naturally incorporated into critical Lifshitz-type theories, it remains unclear how to describe more general cases

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