Gauging Modulated Symmetries

Gui Delfino

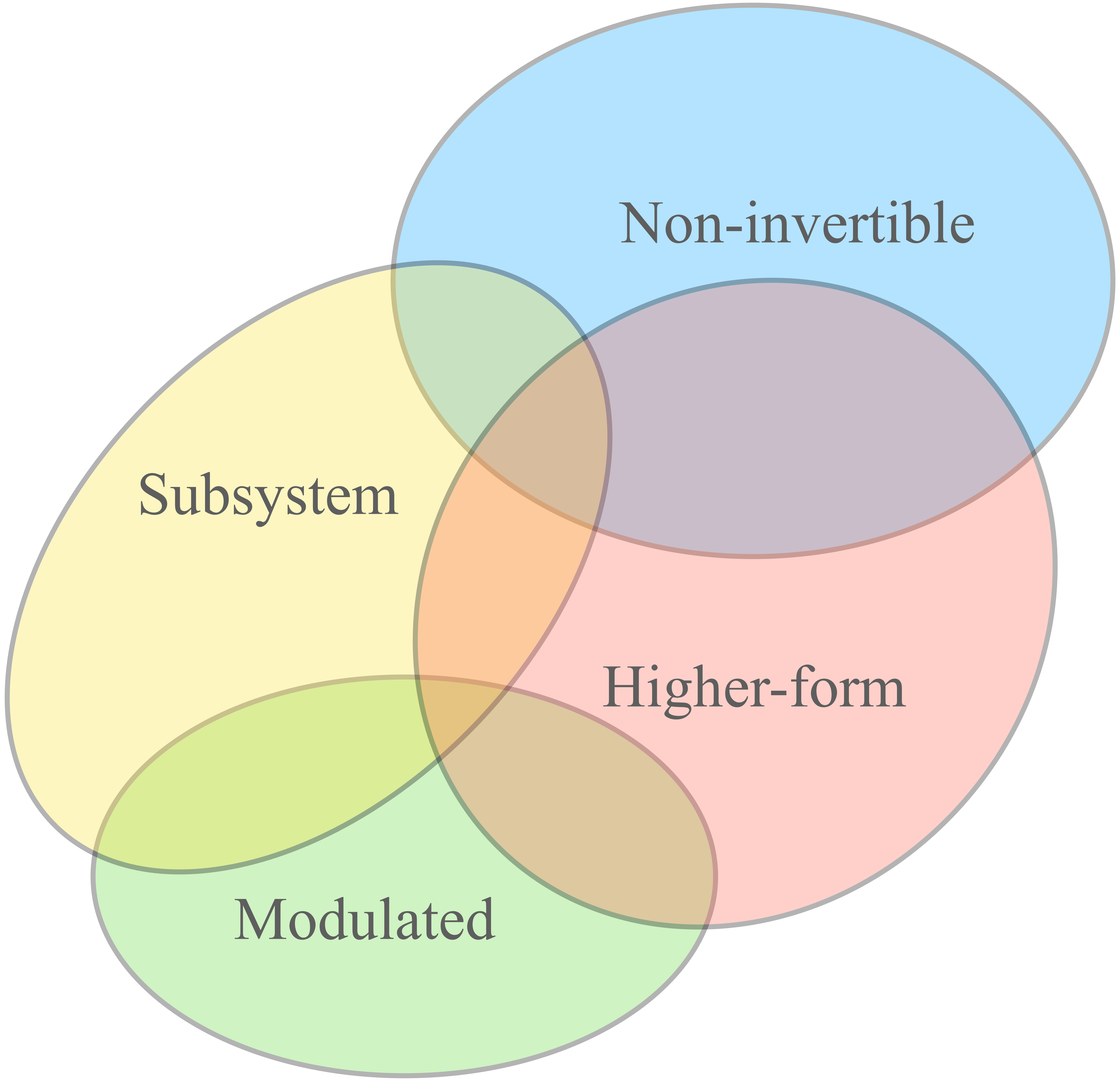
PURDUE
UNIVERSITY

U.S. DEPARTMENT OF
ENERGY
Office of Science

BOSTON
UNIVERSITY

Introduction

- Generalized symmetries extend the conventional notion of symmetry in several distinct ways:
  - Fusion ring multiplication rules (**non-invertible**)
  - Support on topological sub-manifolds (**higher-form**)
  - Support on rigid subspaces (**subsystem**)
  - Non-uniform in space (**modulated**)



Introduction

- Typically, internal and space symmetry groups G_{int} and G_{space} do not talk to each other
- Spatially **modulated symmetries**, on the other hand, act non-uniformly in space and have a non-trivial interplay with space-like symmetries
- This is incorporated through a **group homomorphism** which defines a non-trivial action $\varphi : G_{\text{space}} \to \text{Aut}(G_{\text{int}})$
- Total symmetry group $G = G_{\text{int}} \rtimes_{\varphi} G_{\text{space}}$
- In general G_{int} can be either finite or infinite, Abelian or non-Abelian, 0-form or higher form.

Dipole Moment

Consider $G_{\text{int}} = U(1) \times U(1)^d$ generated by **dipole moment** and **charge** in (d+1) dimensions

$$\boxed{U_q = \exp\left(\mathrm{i}\,\alpha_0\, Q\right), \quad \text{and} \quad U^{(i)}_{dip} = \exp\left(\mathrm{i}\,\alpha_i\, D_i\right) \quad \text{with} \quad Q = \int d\mathbf{x}\, \rho(\mathbf{x}) \quad \text{and} \quad D_i = \int d\mathbf{x}\, \mathrm{x}_i\, \rho(\mathbf{x})}$$

- Translations $T_{\mathbf{a}} : \mathbf{x} \to \mathbf{x} + \mathbf{a}$ have a non-trivial action $T_{\mathbf{a}}\, U^{(i)}_{dip}\, T^{\dagger}_{\mathbf{a}} = U^{(i)}_{dip}\, U^{\mathrm{a}_i}_{q}$

Total symmetry group $G = (U(1) \times U(1)^d) \rtimes_{\varphi} G_{\text{space}}$

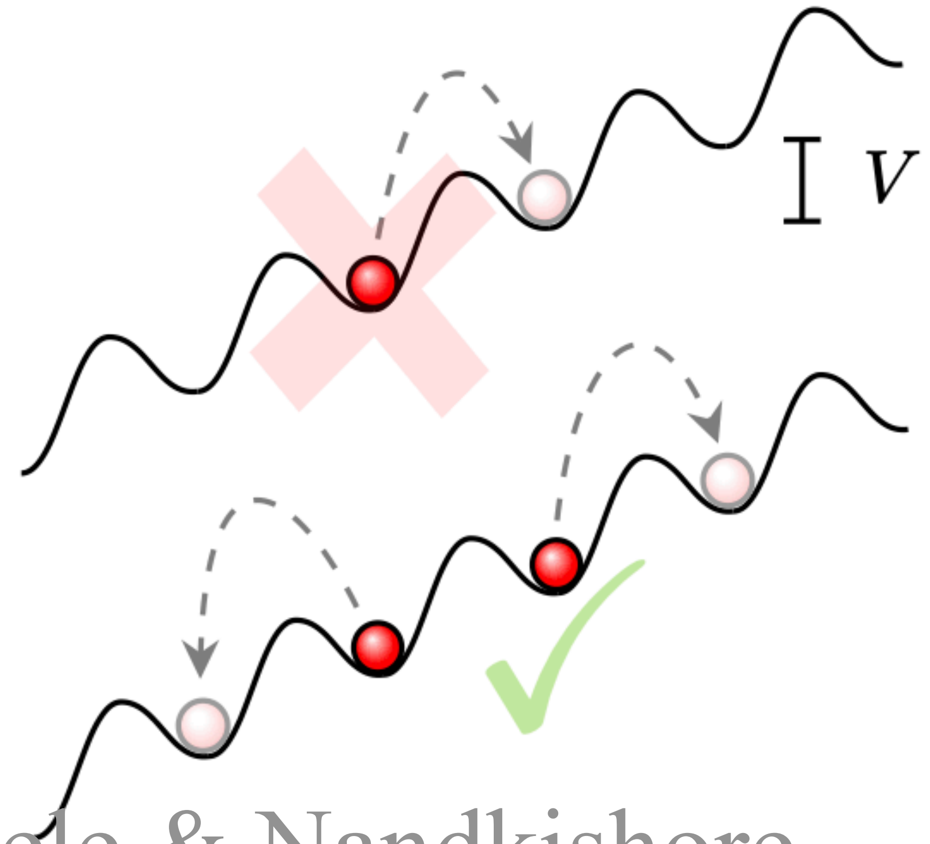
$\Rightarrow$ More generally, the modulation can be implemented by **arbitrary functions**

$$\boxed{G^{(q)} = \int d\mathbf{x}\, f^{(q)}(\mathbf{x})\, \rho(\mathbf{x}),} \quad q = 1, \ldots, n, \quad \text{with} \quad f^{(q)} : \text{space} \to \text{target space}$$

Motivation

Spatially modulated symmetries play an important role in systems with **multicritical points**

~ $z \neq 1$ dynamical critical exponents (Lifshitz, 1941)



- **constrained dynamics:**

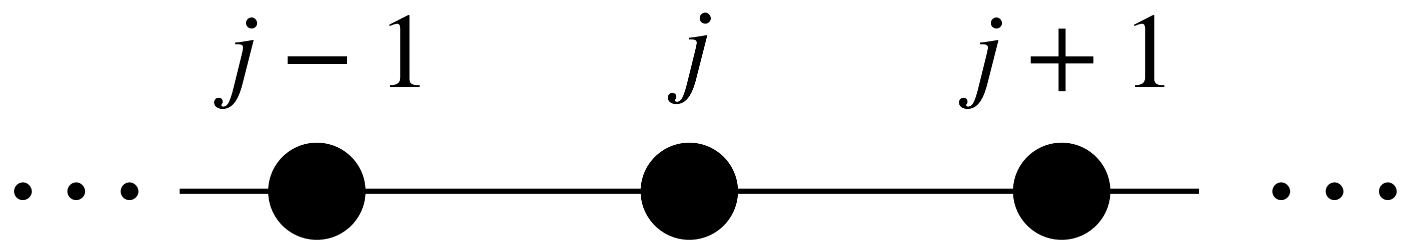
~ Fractons (Pretko, 2017); Hilbert space fragmentation (Khemani, Hermele & Nandkishore, 2020); tilted lattices (Lake, Lee, Han & Senthil, 2023)

- This talk: **Route for crystalline SETs**

~ We focus on lattice systems with G_{int} Abelian and finite

Warm-up: 1D chain

- To warm-up, let us consider one-dimensional lattice chains Λ



- On each site there is a $\mathbb{Z}_N$ **degree of freedom** (*qudit*) $\mathscr{H}_j \cong \mathbb{C}^N$

$$Z_j X_k = e^{\frac{2\pi \mathrm{i}}{N}\delta_{jk}} X_k Z_j \qquad \text{and} \qquad X_j^N = Z_j^N = 1$$

- We consider **finite Abelian groups**, which constrains the functions in symmetry operators

$$\boxed{U_f = \prod_j X_j^{f_j}} \qquad \text{to be} \qquad f : \Lambda \to \mathbb{Z}/N\mathbb{Z}$$

Spatially Modulated Symmetries

The symmetry $U_f = \prod_j X_j^{f_j}$ acts on local operators as

$$Z_j \mapsto e^{\frac{2\pi i}{N} f_j} Z_j \qquad \text{and} \quad X_j \mapsto X_j$$

Takeaway: **The symmetry acts differently in different points in space**

- We want to preserve crystalline symmetries ⇒ one must conserve not only U_f but also $U_{g(f)}$ where $g(f)$ is the function f transformed by crystalline action $g \in G_{\text{space}}$
- Thus, the set of operator symmetries $U_q = \prod_j X_j^{f_j^{(q)}}$ must **close under G_{space}**. We denote the set of modulating functions $S = \{f^{(q)}\}_{q=1,\ldots,r}$.

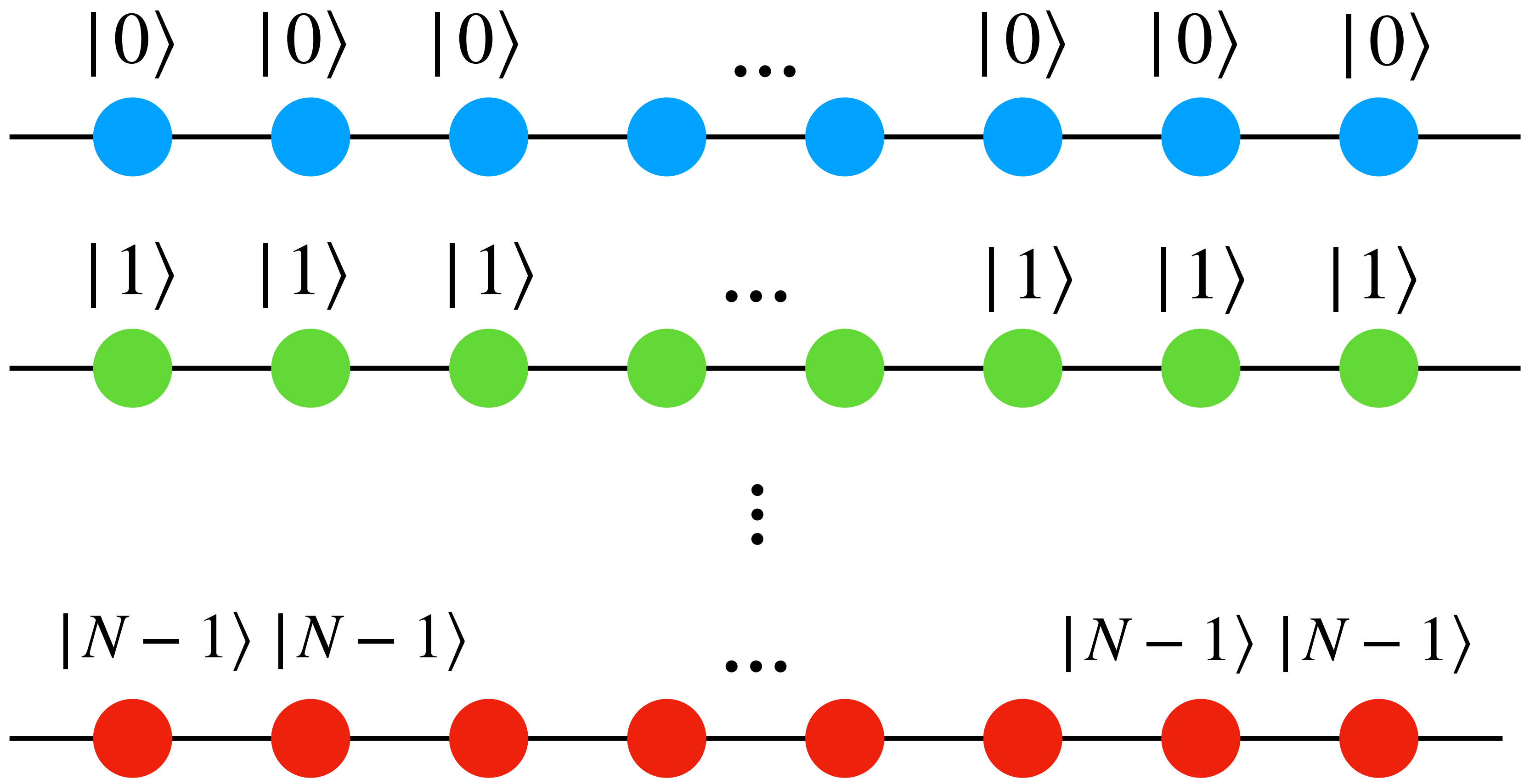
$\mathbb{Z}_N$ Ising in 1+1D

- Prototypical example: Hamiltonian that is invariant under U for a uniform $f_j = 1$

$$H(g) = -\sum_j Z^\dagger_{j-1} Z_j - g \sum_j X_j + \text{h.c.}$$

From $U = \prod_j X_j$, it follows that $UHU^\dagger = H$

Internal group $G_{int} = \mathbb{Z}_N$ with trivial action under translations $TUT^\dagger = U$. Thus $G = \mathbb{Z}_N \times \mathbb{Z}$

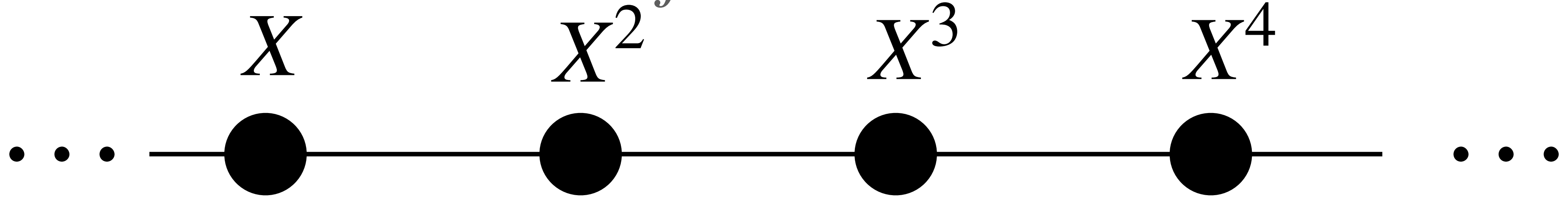


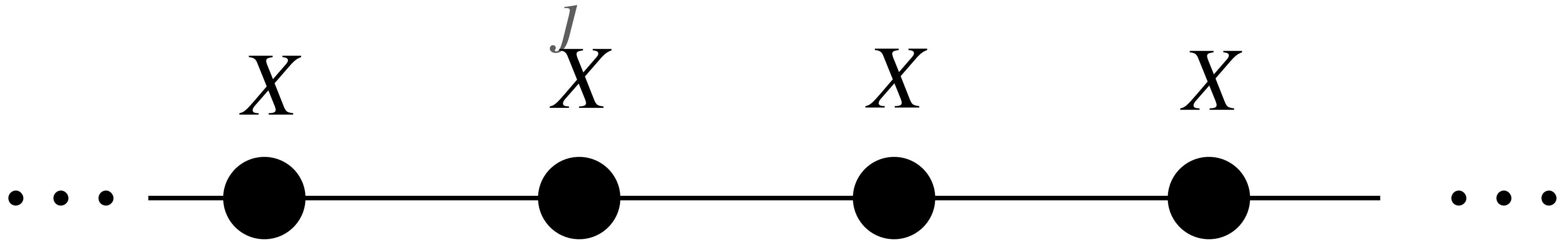
- In 1D, in the $g \ll 1$ limit, SSB and N-fold degenerate ground states with $\langle Z_i^\dagger Z_j \rangle \sim 1$

$\mathbb{Z}_N$ Dipole in 1+1D

- Consider now "$\mathbb{Z}_N$ **dipole moment**" $f_j = j$

$$D = \prod_j X_j^j \quad (\text{“dipole”}) \qquad\qquad Q = \prod_j X_j \quad (\text{“charge”})$$





- $D^N = Q^N = I \quad \Rightarrow \quad G_{int} = \mathbb{Z}_N \times \mathbb{Z}_N$

- Both these operators commute with $H(g)$ and obey

$$H(g) = -\sum_j Z_{j-1} Z_j^{\dagger\,2} Z_{j+1} - g \sum_j X_j + \text{h.c.}$$

Structure of Modulation

- **Translations** $T : j \to j+1$ implies an homomorphism which encodes

$$T\,D\,T^\dagger = D\,Q$$

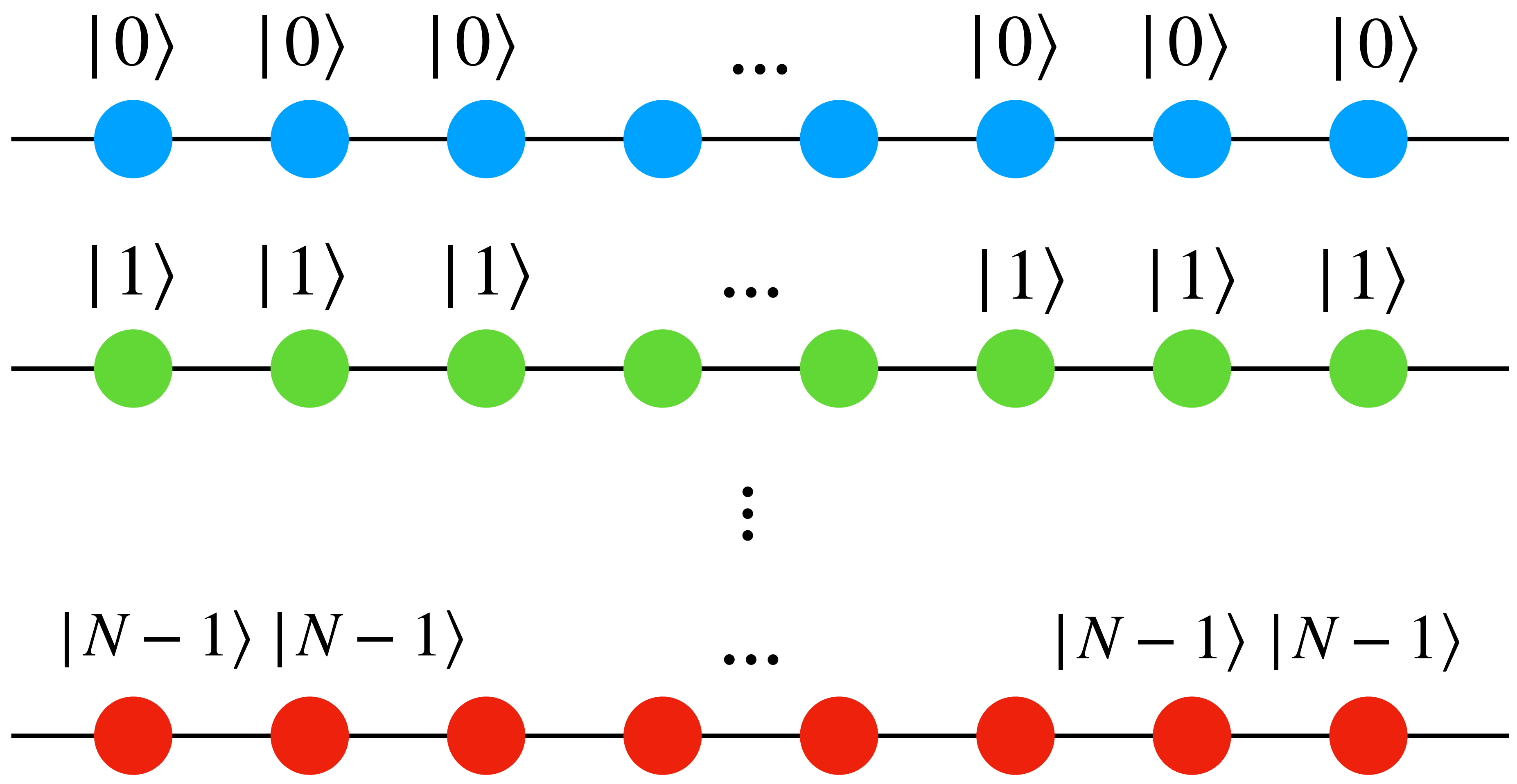
$\sim \varphi(T)$ is an **automorphism** in $\mathbb{Z}_N \times \mathbb{Z}_N$ that maps the **symmetry generators**

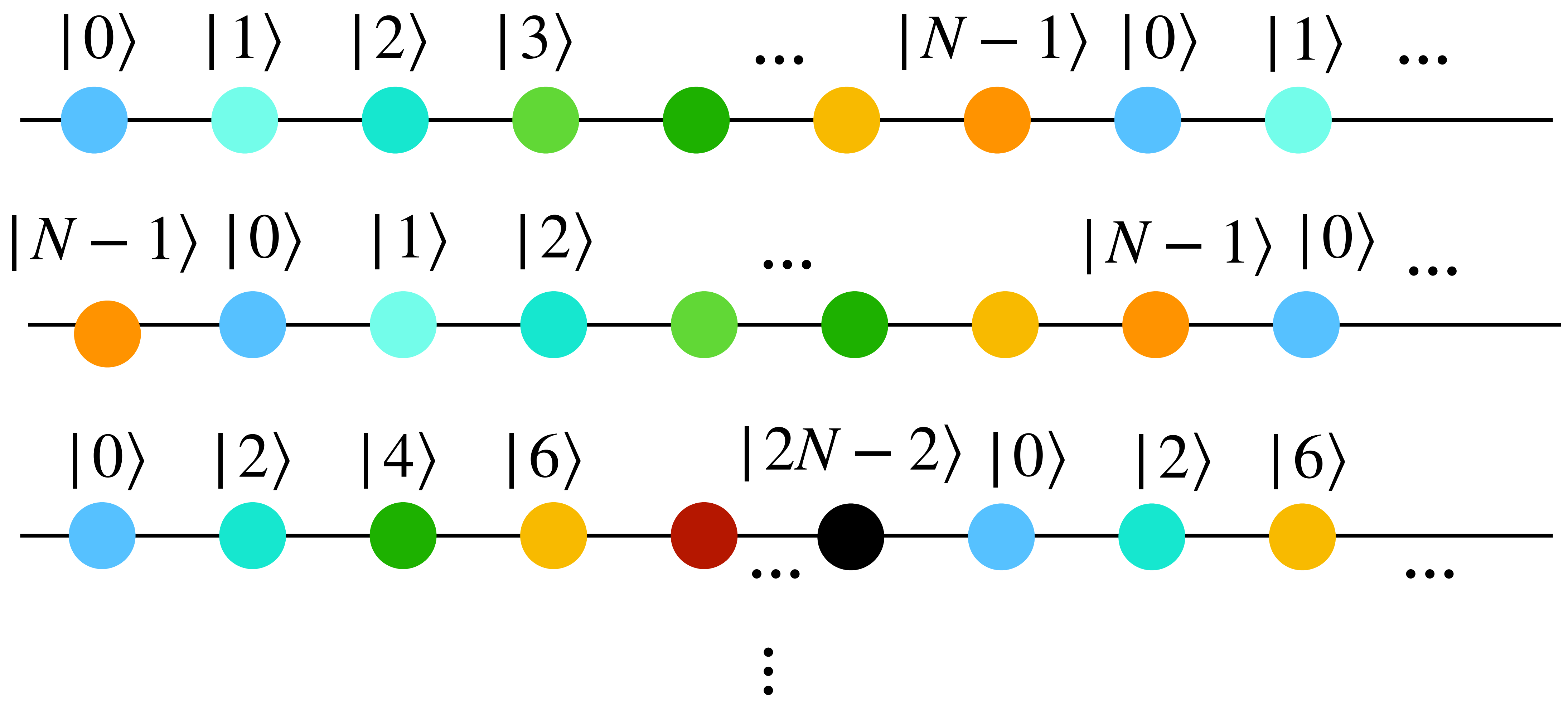
$$\varphi(T) : Q \mapsto Q \qquad \text{and} \qquad \varphi(T) : D \mapsto QD$$

$\Rightarrow$ Total symmetry group $G = (\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes_\varphi \mathbb{Z}$

Dipole Symmetry SSB

- SSB of G_{int} ($h \ll 1$ limit):





There are N^2 degenerate ground states, which transform among themselves under translations

⇒ Spatially modulated degenerate ground states (Lake, Hermele & Senthil, 2022)

Examples in (2+1)D

Let $\mathbf{x} = (x, y) \in \mathbb{Z} \times \mathbb{Z}$ label the coordinates of a two-dimensional lattice. For oncreteness, consider square lattice $G_{\text{space}} = C_4 \rtimes \mathbb{Z}^2$

- Examples of **modulating functions** (which transform according to Irreps of point-groups):

$f_{\mathbf{x}} = 1,$	Charge
$f_{\mathbf{x}} = x, y \mod N$	Dipole
$f_{\mathbf{x}} = x^2 - y^2, xy \mod N$	Quadrupole
$\vdots$	
$f_{\mathbf{x}} = \text{degree n poly} \mod N$	2^n-Multipole
$\vdots$	
$f_{\mathbf{x}} = a^{x+y} \mod N, \quad a \in \mathbb{Z}$	Exponentially modulated charge

(Gromov, 2019; Sala, Lehmann, Rakovszky & Pollmann, 2022)

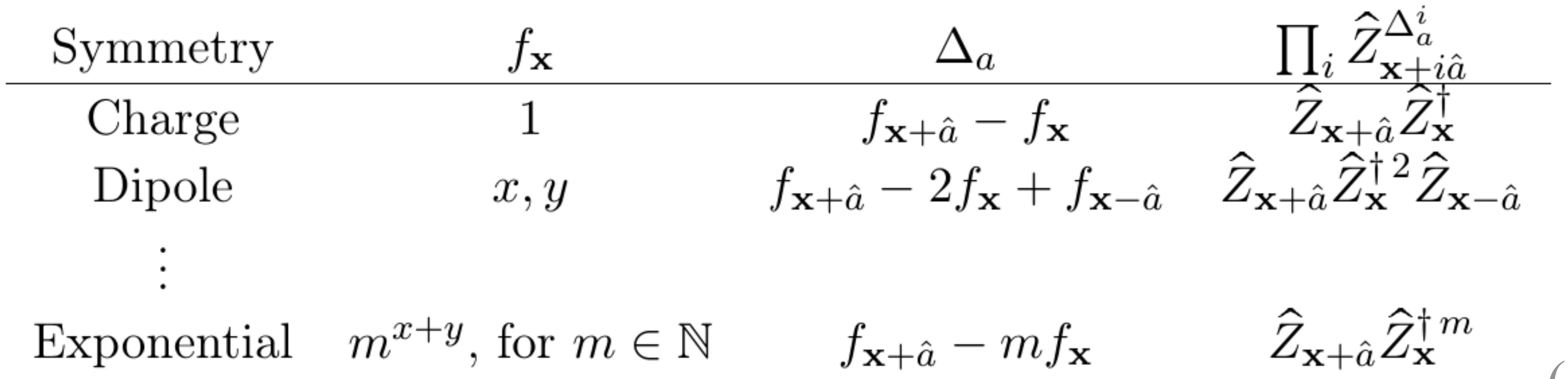
Symmetric Hamiltonians

Let $S = \{f^{(q)}\}_{q=1,\dots,r}$ be the set of functions such that $U_q = \prod_j X_j^{f_j^{(q)}}$ are symmetries. Symmetric Hamiltonians:

$$H_{\text{matter}} = -t \sum_{\mathbf{x},a} \prod_i Z_{\mathbf{x}+i\hat{a}}^{\Delta_a^i} + \sum_{\mathbf{x}} X_{\mathbf{x}} + \mathrm{h.c.} + \dots$$

where the Δ_a^i are $\mathbb{Z}_N$ coefficients that define a "**lattice derivative**" operator $\sum_i \Delta_a^i f_{\mathbf{x}+i\hat{a}} \equiv \Delta_a f$

- The coefficients Δ_a^i are determined by the condition $\Delta_a f^{(q)} = 0$, for $1 = 2,\dots,r$.
- Δ_a defines $\mathcal{O}(r)$-body interactions ⇒ non-periodic functions require non-local Hamiltonians!



Symmetry	$f_{\mathbf{x}}$	Δ_a	$\prod_i \hat{Z}^{\Delta_a^i}_{\mathbf{x}+i\hat{a}}$
Charge	1	$f_{\mathbf{x}+\hat{a}} - f_{\mathbf{x}}$	$\hat{Z}_{\mathbf{x}+\hat{a}}\hat{Z}^{\dagger}_{\mathbf{x}}$
Dipole	x, y	$f_{\mathbf{x}+\hat{a}} - 2f_{\mathbf{x}} + f_{\mathbf{x}-\hat{a}}$	$\hat{Z}_{\mathbf{x}+\hat{a}}\hat{Z}^{\dagger\,2}_{\mathbf{x}}\hat{Z}_{\mathbf{x}-\hat{a}}$
$\vdots$			
Exponential	m^{x+y}, for $m \in \mathbb{N}$	$f_{\mathbf{x}+\hat{a}} - m f_{\mathbf{x}}$	$\hat{Z}_{\mathbf{x}+\hat{a}}\hat{Z}^{\dagger\,m}_{\mathbf{x}}$

(GD, Chamon & You, 2023)

SETs: Beyond SSB

Gauging modulated symmetries in 2+1D is a mechanism to get topologically ordered with locally constrained anyons

Deconfined phase of gauge theory ↔ Topological Order

The resulting topological orders are enriched by G_{space} and correspond to **crystalline symmetries enriched topological order** (SETs)

Under lattice symmetry actions $g \in G_{\text{space}}$, anyons permute flavors

$$g : a \mapsto {}^{g}a$$

so that topological data is preserved.

Gauging G_{int}

- Gauging G_{int} corresponds to a two step process:

1. I) **Extend Hilbert space** by introducing $\mathbb{Z}_N$ gauge fields A_a and E_a
2. II) **Impose Gauss law**, generator of gauge transformations

$$Z_{\mathbf{x}} \to e^{i\alpha_{\mathbf{x}}} Z_{\mathbf{x}} \qquad \text{and} \qquad A_{a,\mathbf{x}} \to A_{a,\mathbf{x}} + \Delta_a \alpha_{\mathbf{x}}$$

- Minimally coupled theory

$$H_{\text{matter+gauge}} = -\sum_{\mathbf{x},a} e^{-iA_{a,\mathbf{x}}} \prod_i \hat{Z}^{\Delta_a^i}_{\mathbf{x}+i\hat{a}} + \text{h.c.} + \ldots$$

(2+1)D Modulated Gauge Theories

- From the canonical relation $[A_{a,\mathbf{x}}, E_{b,\mathbf{y}}] = i\delta_{a,b}\delta_{\mathbf{x},\mathbf{y}}$, gauge transformations define a **Gauss law**

$$q_{\mathbf{x}} = \Delta_a E_a\,.$$

- One can also define a gauge invariant **magnetic flux**, schematically expressed as

$$b_{\mathbf{x}} = \check{\Delta}_a A_a$$

where $\check{\Delta}_a$ is defined such that $\check{\Delta}_a \Delta_a \alpha = 0$ for any lattice function $\alpha_{\mathbf{x}}$, which enforces that $b_{\mathbf{x}}$ is gauge invariant.

- Δ_a and $\check{\Delta}_a$ depend on the functions $f_{\mathbf{x}}$ and the lattice symmetries one wishes to preserve (upcoming work)

Deconfined phase

- Introduce $\mathbb{Z}_N$ star and plaquette operators with $X_a \sim e^{2\pi i E_a/N}$, and $Z_a \sim e^{2\pi i A_a/N}$

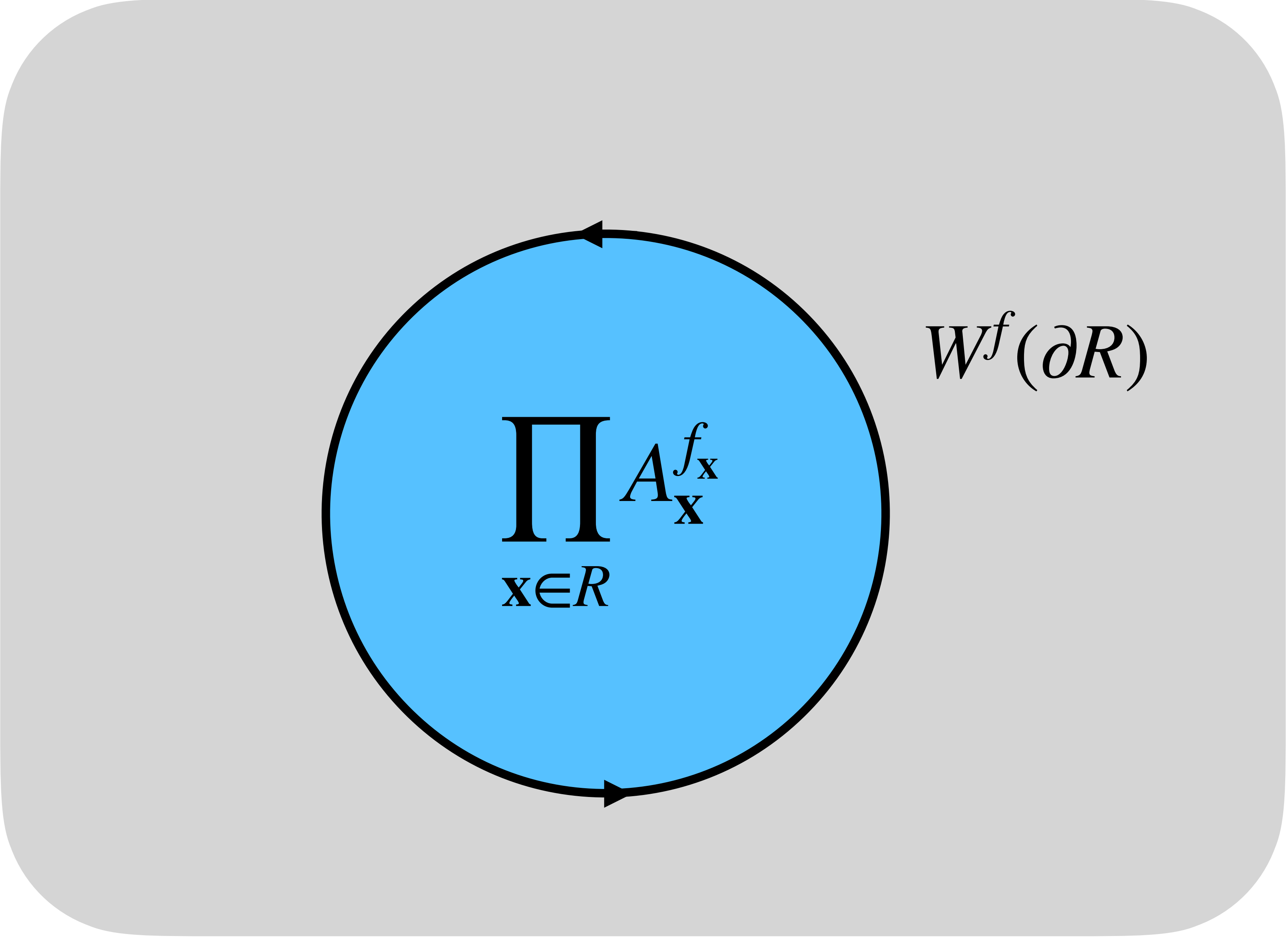
$$A_{\mathbf{x}} = e^{\frac{2\pi i q_{\mathbf{x}}}{N}} = \prod_a X_a^{\Delta_a} \qquad \text{and} \qquad B_{\mathbf{x}} = e^{\frac{i b_{\mathbf{x}}}{N}} = \prod_a Z_a^{\check{\Delta}_a}$$

- And let $\tilde{f} \in \mathrm{Ker}(\check{\Delta})$, under periodic boundary conditions $\prod_{\mathbf{x}\in\Lambda} A_{\mathbf{x}}^{f_{\mathbf{x}}} = 1$, and $\prod_{\mathbf{x}\in\Lambda} B_{\mathbf{x}}^{\tilde{f}_{\mathbf{x}}} = 1.$

Finally, for ∂R be the boundary of a region R subset of Λ, then

$$\prod_{\mathbf{x}\in R} A_{\mathbf{x}}^{f_{\mathbf{x}}} = W^f(\partial R) \quad \text{and} \quad \prod_{\mathbf{x}\in R} B_{\mathbf{x}}^{\tilde{f}_{\mathbf{x}}} = V^{\tilde{f}}(\partial R),$$

define **1-form symmetries** W^f and $V^{\tilde{f}}$ of the gauge theory



Modulated 1-form symmetries

- The higher-form symmetries W^f and $V^{\tilde{f}}$ depend explicitly on the functions f and $\tilde{f}$ and are **modulated**

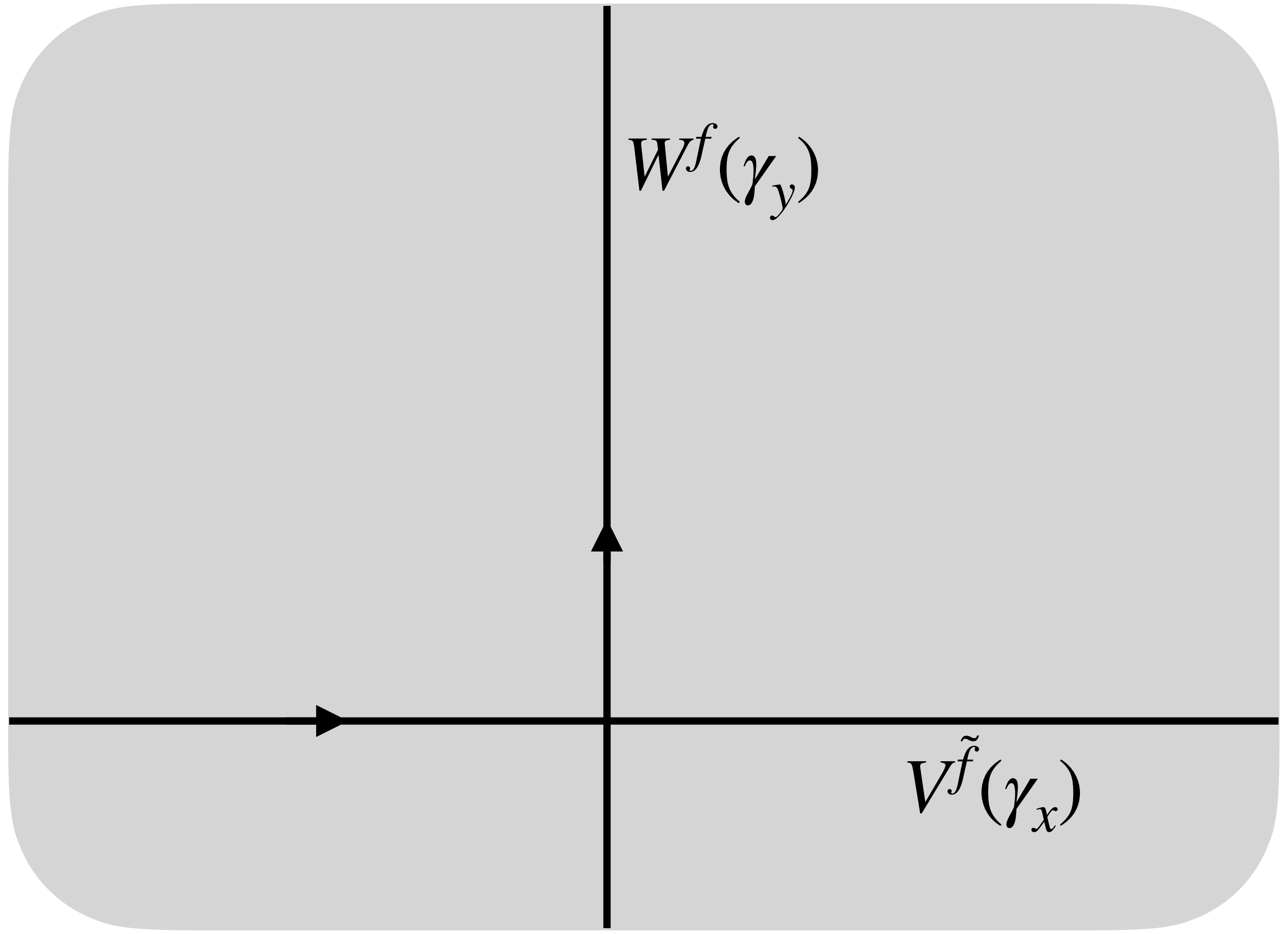
$$TW^fT^\dagger = W^{g(f)}$$

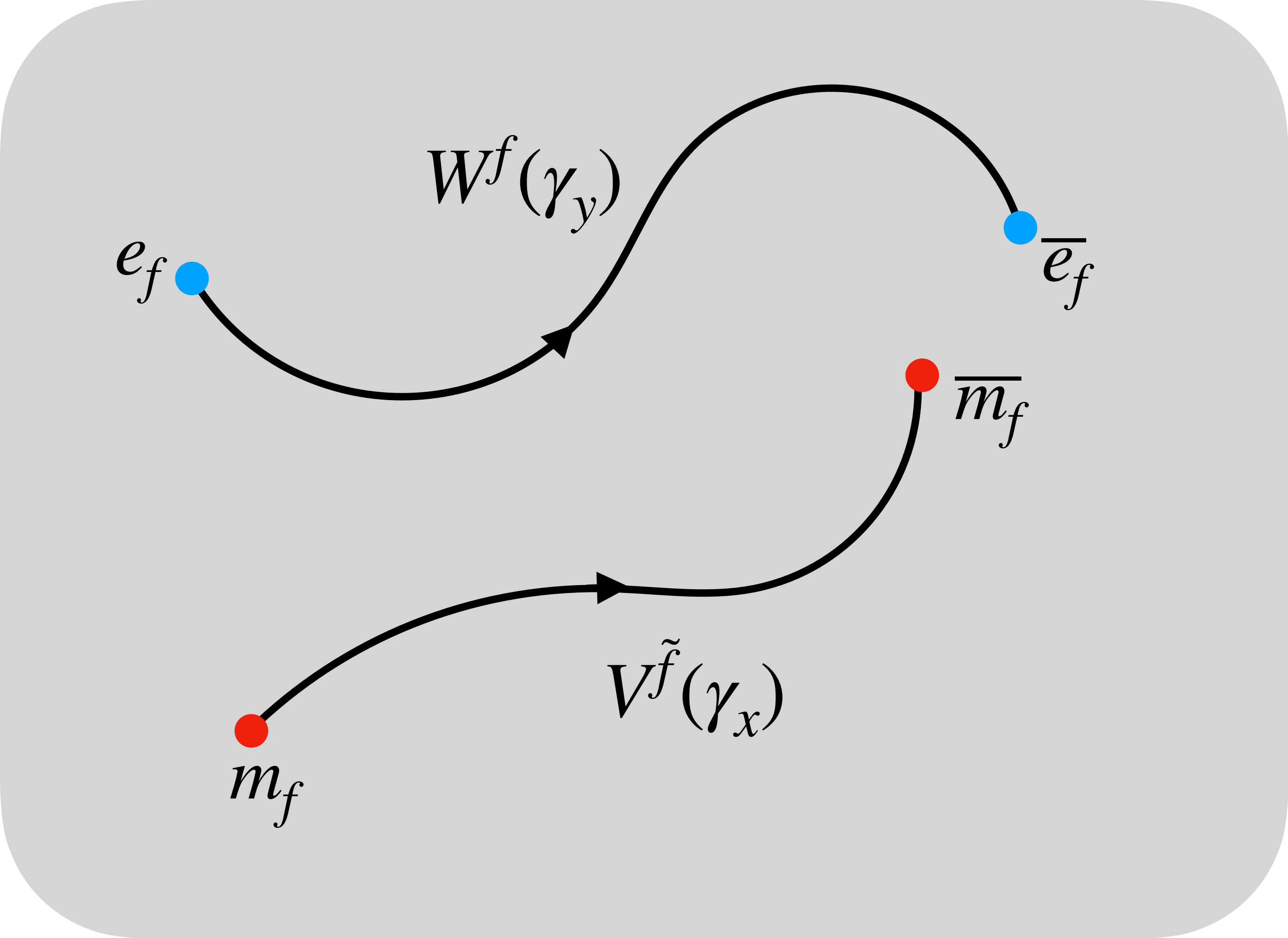
- Additionally, they share a **mixed anomaly**

$$W^f(\gamma)V^{\tilde{f}}(\gamma^*) = \omega^{I(\gamma,\gamma^*)}V^{\tilde{f}}(\gamma^*)W^f(\gamma),$$

where $I(\gamma, \gamma^*)$ counts the lattice intersection and $\omega = e^{2\pi i/N}$.

- Open strings $W^f(\gamma)$ and $V^f(\gamma^*)$ create **Abelian anyonic excitations** at the endpoints of γ and γ^*





(1+1)D Gauging

In 1+1D the story is a bit simpler (translations only):

$$G = G_{\text{int}} \rtimes_{\varphi} G_{\text{space}} \quad \overset{gauge\ G_{int}}{\Rightarrow} \quad G^{\vee} = G_{\text{int}} \rtimes_{\varphi^{\vee}} G_{\text{space}}$$

- In general, the symmetry can have different modulation φ and $\varphi^{\vee}$

~ For the case in which φ and φ are isomorphic, the theories **self-dual** under gauging (KW-like duality);

~ We also find **KW non-invertible symmetries**, as well as their dependence on the modulation structure;

(Pace, GD, Lam & Aksoy, 2025)

(2+1)D Gauging

- This procedure provide us exactly solvable lattice models for crystalline SETs

$$G = G_{\text{int}} \rtimes_{\varphi} G_{\text{space}} \quad \overset{gauge\ G_{int}}{\Rightarrow} \quad G^{\vee} = \left(G^{(1)}_{\text{int}} \times G^{(1)}_{\text{int}}\right) \rtimes_{\varphi^{\vee}} G_{\text{space}}$$

Examples:

$\mathbb{Z}_N$ Rank 2 Toric Code $\leftrightarrow G_{\text{int}} = \mathbb{Z}_N^3$ with $f_{\mathbf{x}} = 1, x, y$

(Oh, Kim, Moon & Han, 2022)

$\mathbb{Z}_N$ dipolar-quadrupolar code $\leftrightarrow G_{\text{int}} = \mathbb{Z}_N^4$ with $f_{\mathbf{x}} = 1, x, y, xy$

(GD, Fontana, Gomes & Chamon, 2023)

Crystalline SETs

- Anyons have **restricted mobility**, incorporated in the Gauss law and magnetic flux through Δ and $\check{\Delta}$
- Ground state degeneracy depends on system size (flavor of **UV/IR mixing**)
- Anyons have **position-dependent quantum numbers** (Pace & Wen, 2022)
- Low energy effective theories are equivalent to **twisted TQFT** gauge theories
- Different crystalline SETs can be connected through phase transitions driven by **anyon condensation** (GD & You, 2024)

Subsystem ~ Modulated

- Fracton topological order in (3+1)D is understood in terms of **subsystem symmetries** (Vijay, Haah & Fu, 2016)
- Consider a subsystem symmetry operator $U_\Omega = \prod_{\mathbf{x}\in\Omega} X_{\mathbf{x}}$ with support on a sub-lattice $\Omega \subset \Lambda$ of the whole lattice Λ

(subsystem) $$\prod_{\mathbf{x}\in\Omega} X_{\mathbf{x}} = \prod_{\mathbf{x}\in\Lambda} X_{\mathbf{x}}^{\delta_{\mathbf{x}}(\Omega)}$$ **(modulated)**

$\Rightarrow$ 3D fracton systems can be casted as a modulated gauge theories associated to infinitely many modulated symmetries $f^{(i)}_{\mathbf{x}} = \delta_{\mathbf{x}}(\Omega_i)$ for an extensive number of sub-lattices $\Omega_i \subset \Lambda,\ i = 1,\ldots,L$.

Final Considerations

- Gauging modulated symmetries is cheap way of constructing exactly solvable models for **crystalline SETs**
- Classification and characterization of such phases are mapped into the study of **functions and number theory**;
- Modulated gauge theories can take into account a **whole spectrum of enriched Abelian topological phases**

$f = \text{cte}$	$f = \text{poly}$	$f = \text{non} - \text{poly}$	$f^{(i)} = \delta(\Omega_i)_{i=1,\ldots,L}$	$\ldots$
Topological order		SETs	Fractons	?

Future Perspectives

- Flow into **continuum**: Does modulated symmetries have an exact realization in the continuum?

  ⇒ May offer insights into type-II fracton field theories, with fractal subsystem symmetries

- Investigate the physics of **critical points**: non-CFTs continuous phase transitions?

  ⇒ While polynomial symmetries are naturally incorporated into critical Lifshitz-type theories, it remains unclear how to describe more general cases

⋮

Collaborators


Pedro Gomes


Claudio Chamon


Yizhi You


Weslei Fontana


Salvatore Pace


Ho Tat Lam


Omer Aksoy