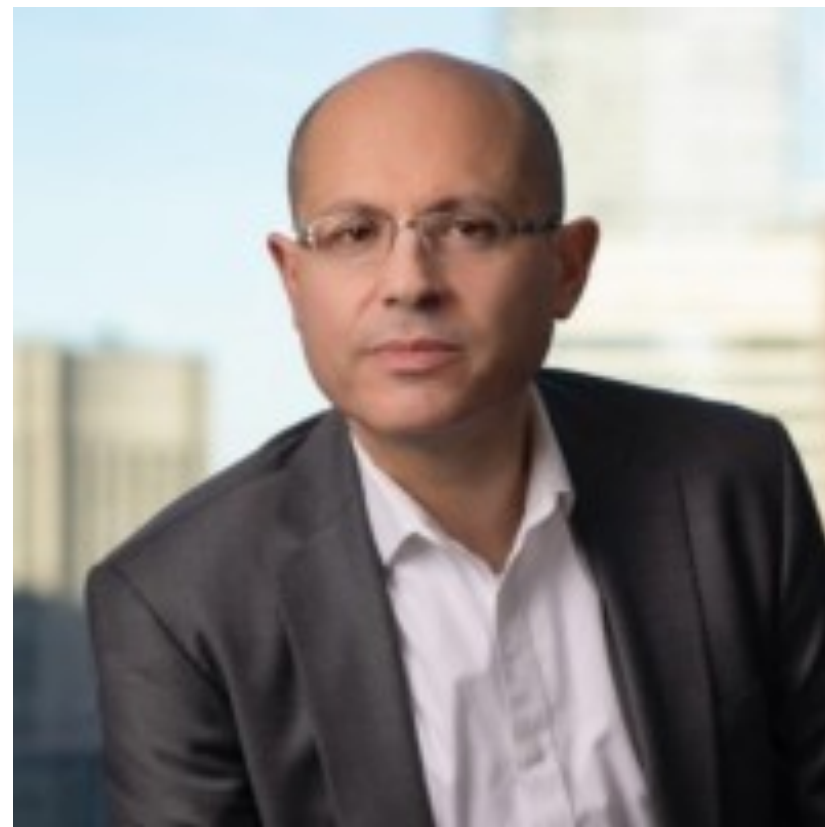


# TOPOLOGICAL META-MATERIALS FROM COOPER-PAIR SPLITTERS

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Gui Delfino  
PRX Quantum 6, 030318 (2025)



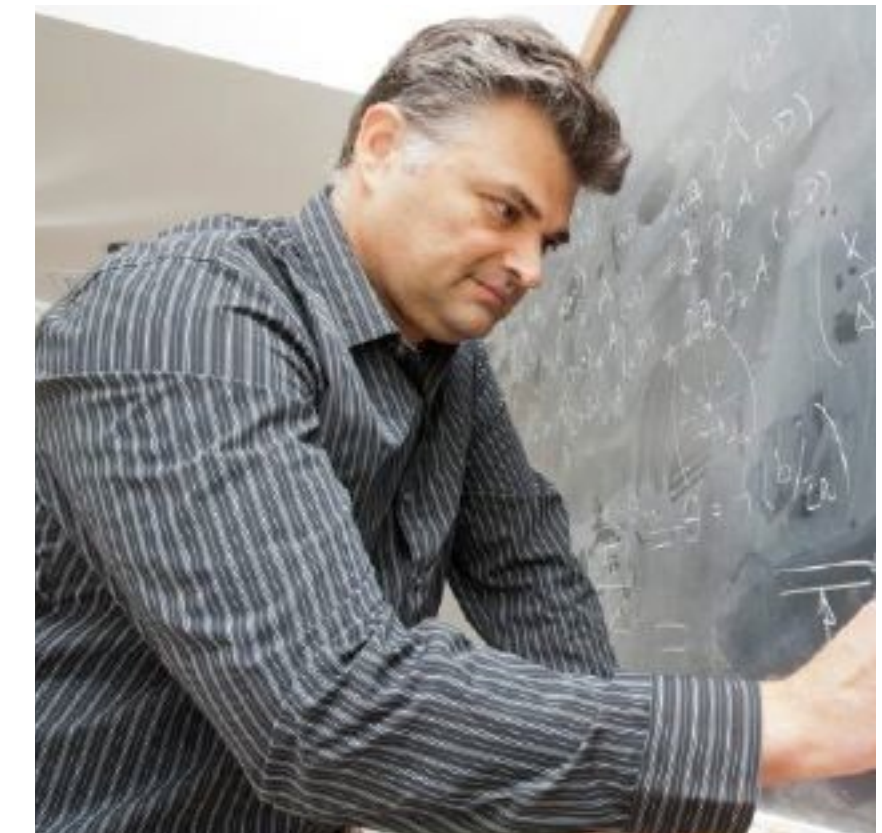
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Charles M. Marcus



Claudio Chamon

# Introduction

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- Topological phases of matter are gapped and highly entangled (spin liquids, topological superconductors, etc.)
- They are hard to realize experimentally!
- Alternative: Artificial “**meta-materials**”
- Our approach: networks of usual **superconducting wires** under magnetic fields
  - ⇒ These networks are built out of smaller blocks: **Cooper-pair splitters**

We find a phase diagram with topological superconductivity

# Topological Superconductors

- Ten-fold classification: classifies free-fermion gapped Hamiltonians based on the presence of symmetries (Altland & Zirnbauer, 1997)
- Topological superconductors preserve particle-hole symmetry (classes C and D)
- This table suggests that  $d = 2$  appears to be the natural dimension to work on;
- Additionally, we must **break time reversal symmetry**

AZ	Symmetry			Dimension							
	T	C	S	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

- T: time-reversal,
- C: particle-hole
- S: chiral  $S = TC$

# Chern numbers in 2D

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- The topological invariant in two-dimensions correspond to **Chern numbers**
- Translation invariant Hamiltonians eigenstates  $|n\mathbf{k}\rangle$  can be labelled by band index  $n$  and crystal momentum  $\mathbf{k}$
- The Chern numbers for the  $n$ -th band can be computed through the sum over the Brillouin zone

$$c_n = \frac{1}{2\pi} \sum_{\mathbf{k} \in BZ} \nabla_{\mathbf{k}} \times \vec{A}_{n,\mathbf{k}}$$

with  $\vec{A}_{n,\mathbf{k}} = -i\langle n\mathbf{k} | \vec{\nabla}_{\mathbf{k}} | n\mathbf{k} \rangle$  is the Berry connection in the  $n$ -th Brillouin zone

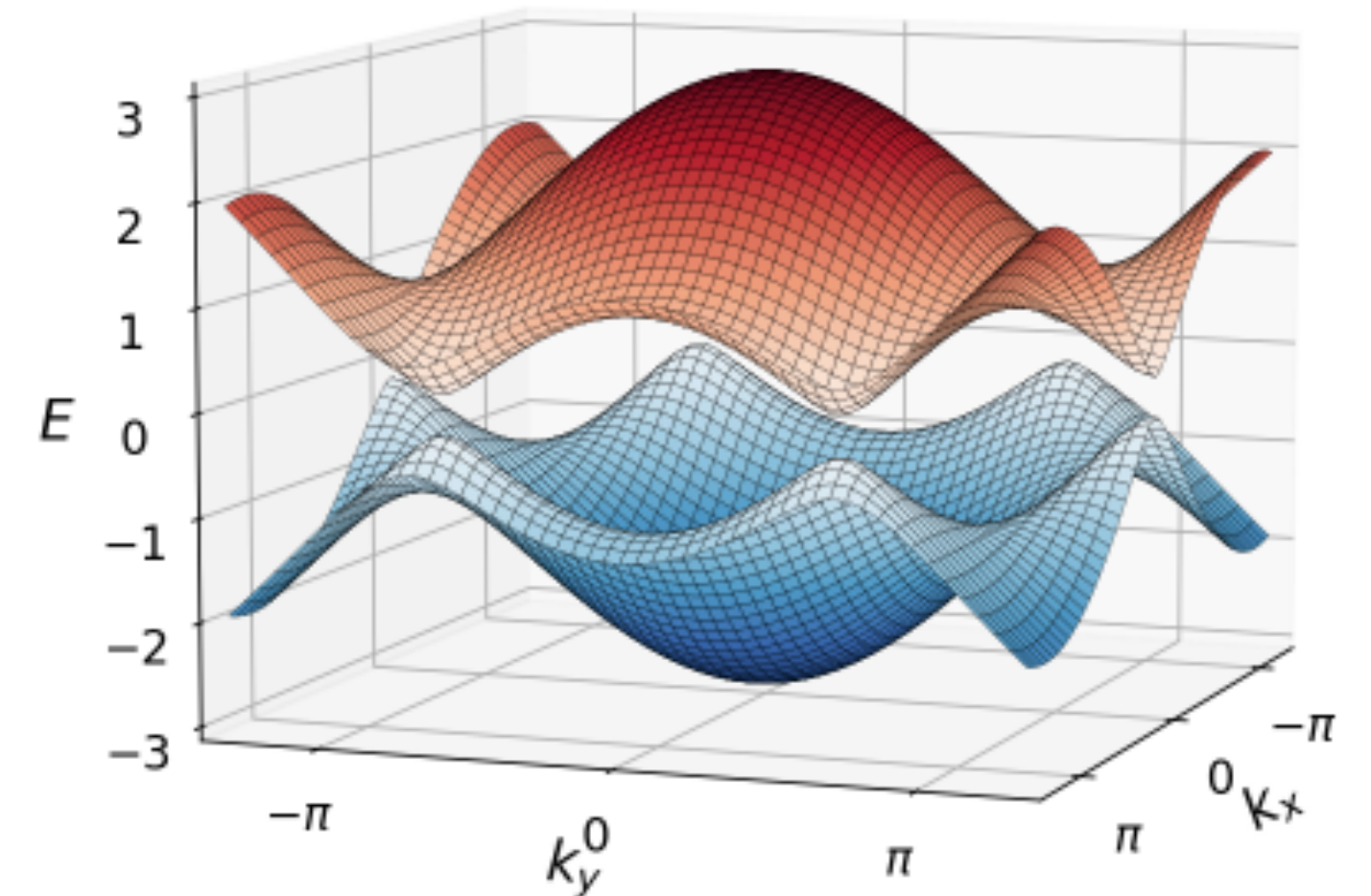
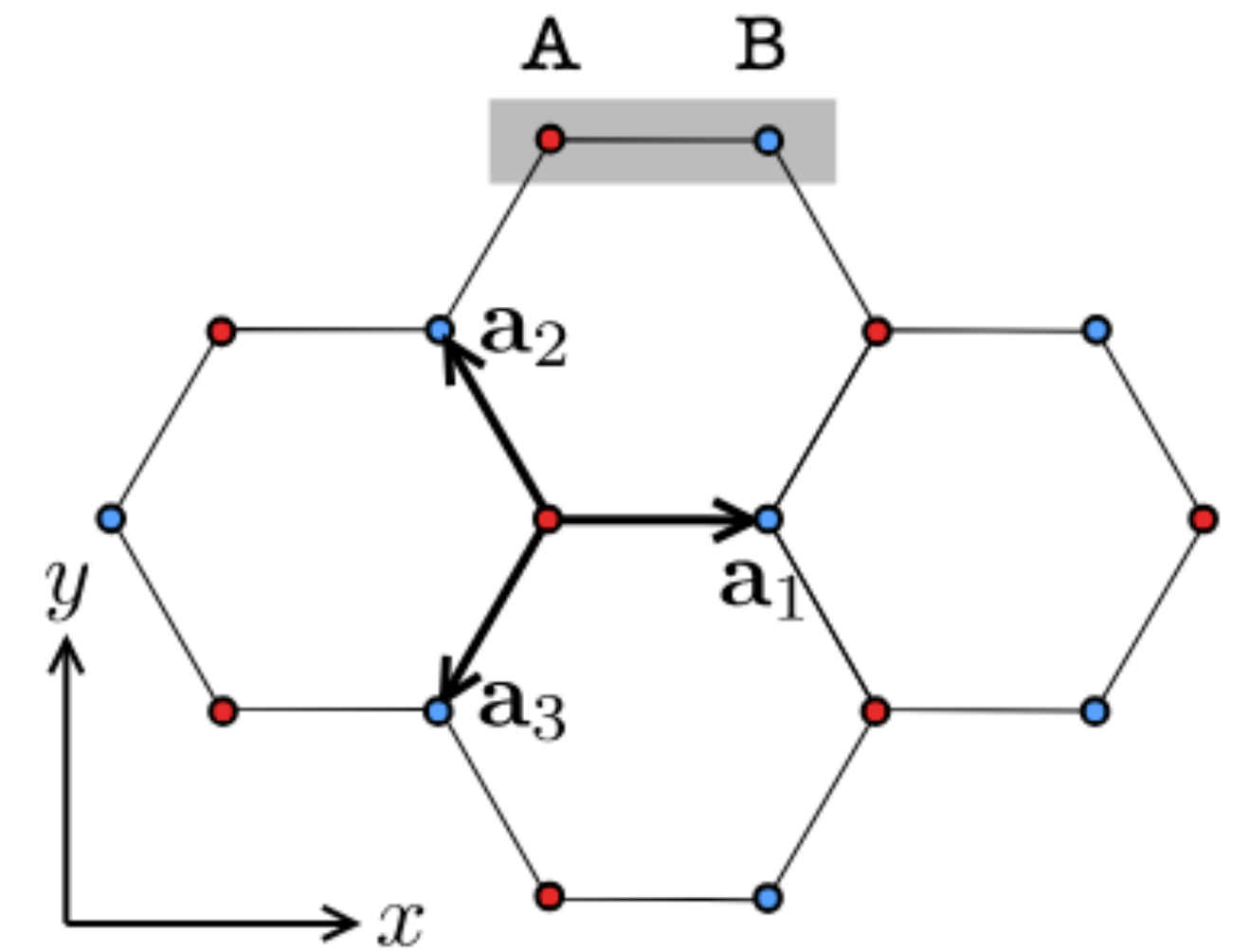
- Chern numbers are **quantized**  $c_n \in \mathbb{Z}$ , constrained to  $\sum_n c_n = 0$

# Example: Chern insulator

- Chern insulators belong to class  $A$ : respect neither  $T$ ,  $C$ , or  $S$ , and in 2D are classified by  $\mathbb{Z}$
- Haldane model (Haldane, 1988)

$$H = t_1 \sum_{\langle i,j \rangle} c_i^\dagger c_j + t_2 \sum_{\langle\langle i,j \rangle\rangle} e^{\pm i\varphi} c_i^\dagger c_j + m \sum_i \epsilon_i c_i^\dagger c_i,$$

- Hopping electrons in a magnetic field  $\varphi = \pi/2$
- In general, non-trivial  $\varphi \pmod{2\pi}$  breaks time-reversal symmetry

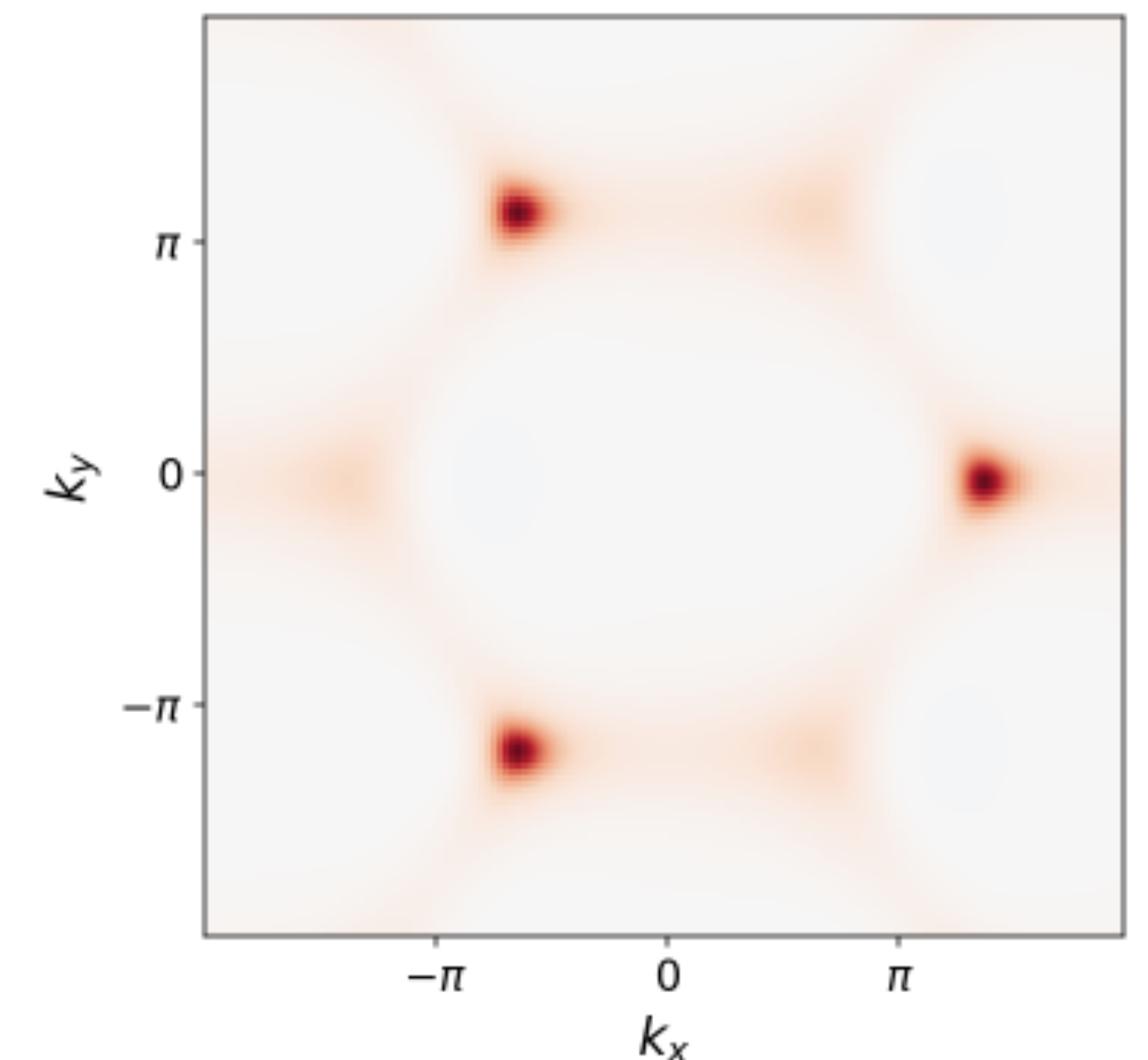
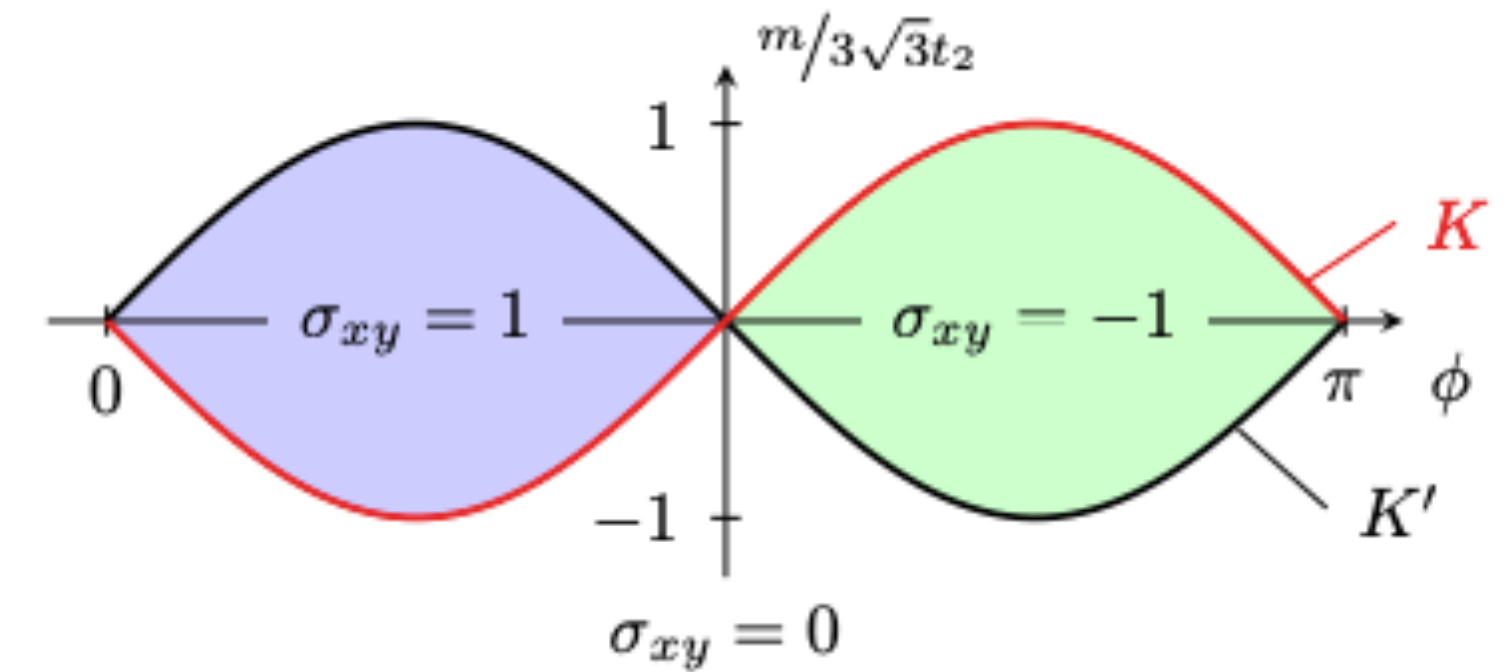


# Chern numbers

- There are three gapped phases which we can access by varying  $t_1, t_2, m$  and  $\varphi$
- Two of them are topological: non-trivial Chern numbers  $\pm 1$
- Chern number is a surface integral of Berry curvature:
  - $\Rightarrow$  analogous of magnetic flux, which measures magnetic sources (monopoles) in Brillouin zone

$$C = \frac{1}{2\pi} \int_{\mathbf{k} \in \text{BZ}} d^2\mathbf{k} \left( \nabla_{\mathbf{k}} \times \vec{A}_{n,\mathbf{k}} \right)$$

- Chern numbers can be probed through quantized Hall response and stable gapless edge modes



# General goal

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- Theoretically **design topological matter** through accessible resources in a lab
  - As underlying degrees of freedom, we are interested in usual, non-topological, **superconducting wires**
  - Couple such wires through **Josephson junctions**
  - Break time-reversal symmetry with **magnetic flux**

# Superconductivity

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- BCS s-wave superconductivity, where  $c_{i\sigma}^\dagger$  is the electron creation operator and  $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$

$$H = -t \sum_{\langle i,j \rangle, \sigma} c_{i\sigma}^\dagger c_{j\sigma} - U \sum_i n_{i\uparrow} n_{i\downarrow}$$

- In **mean field** approach, this reduces to the quadratic Hamiltonian in momentum space

$$\hat{H}_{\text{MF}} = \sum_{\mathbf{k}, \sigma} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left( \Delta c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger + \Delta^* c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right)$$

Where  $\Delta \equiv -U \sum_{\mathbf{k}} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle = |\Delta| e^{i\theta}$  is the **superconductor order parameter** so that

$$1 = U \sum_{\mathbf{k}'} \frac{1}{2E_{\mathbf{k}'}} \tanh\left(\frac{E_{\mathbf{k}'}}{2k_B T}\right) \quad (\text{gap equation})$$

with  $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta|^2}$

# Josephson Junctions

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- The superconductor phase  $\theta$  effectively captures the bosonic character of the condensate
- A Josephson junction (JJ) consists of a **barrier for Cooper pairs** to tunnel through (in fermion description this corresponds to a weak link  $\Gamma$ )



- In the presence of magnetic flux, the SC order parameter shift contributes to the system energy

$$H = -E_J \cos(\theta_1 - \theta_2 - \phi), \quad E_J \sim \frac{\Gamma^2}{|\Delta|}$$

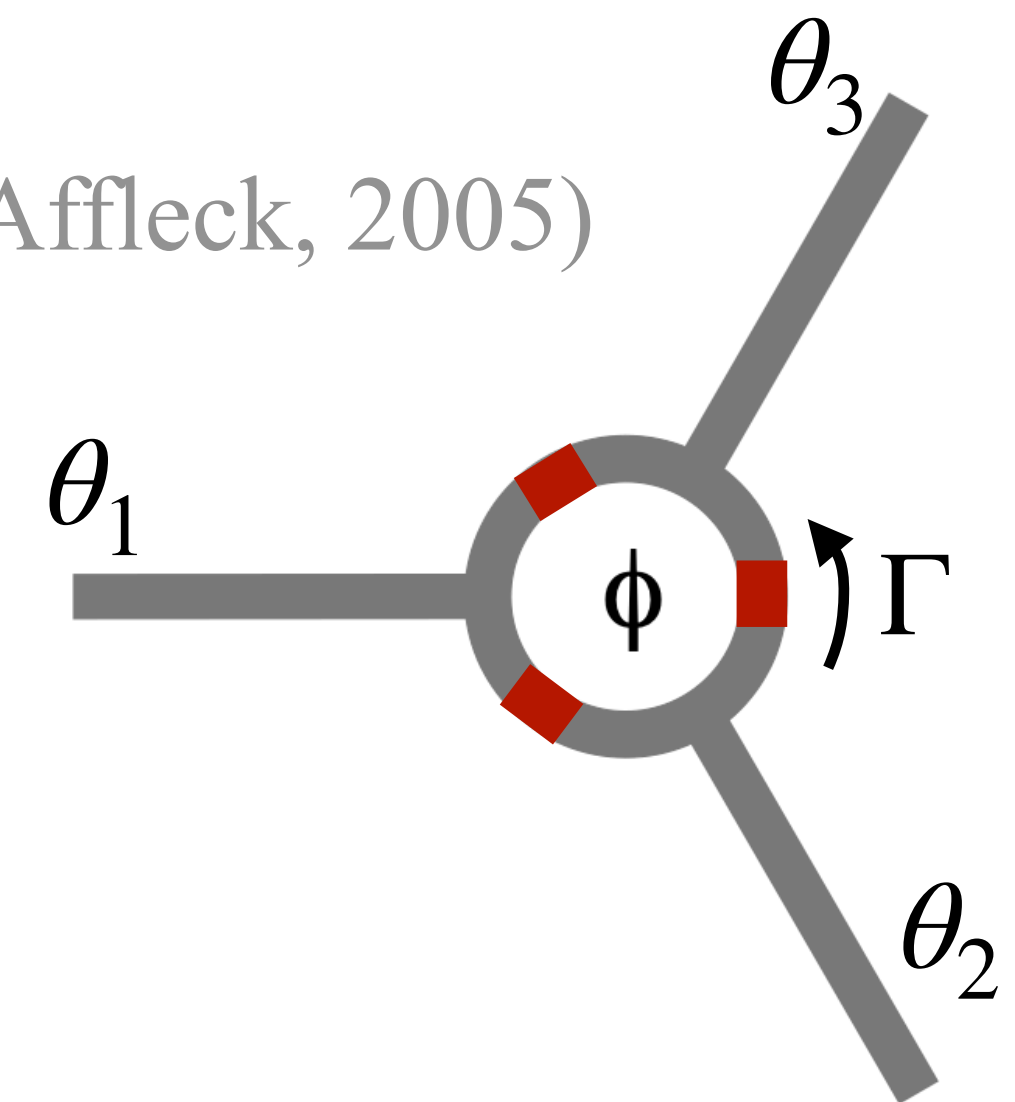
- **Josephson current** through the junction (with no voltage applied)

$$I = I_0 \sin(\theta_1 - \theta_2 - \phi)$$

# Splitters / Circulators

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- We now have the basic ingredients to introduce the **Cooper-pair splitter**
  - Connecting Y-junctions with JJs and adding flux
  - Explored in the context of quantum wires (Oshikawa, Chamon & Affleck, 2005)
- ⇒ Three terminals circulator with flux  $\phi$
- ⇒ Josephson Junctions  $\Gamma$



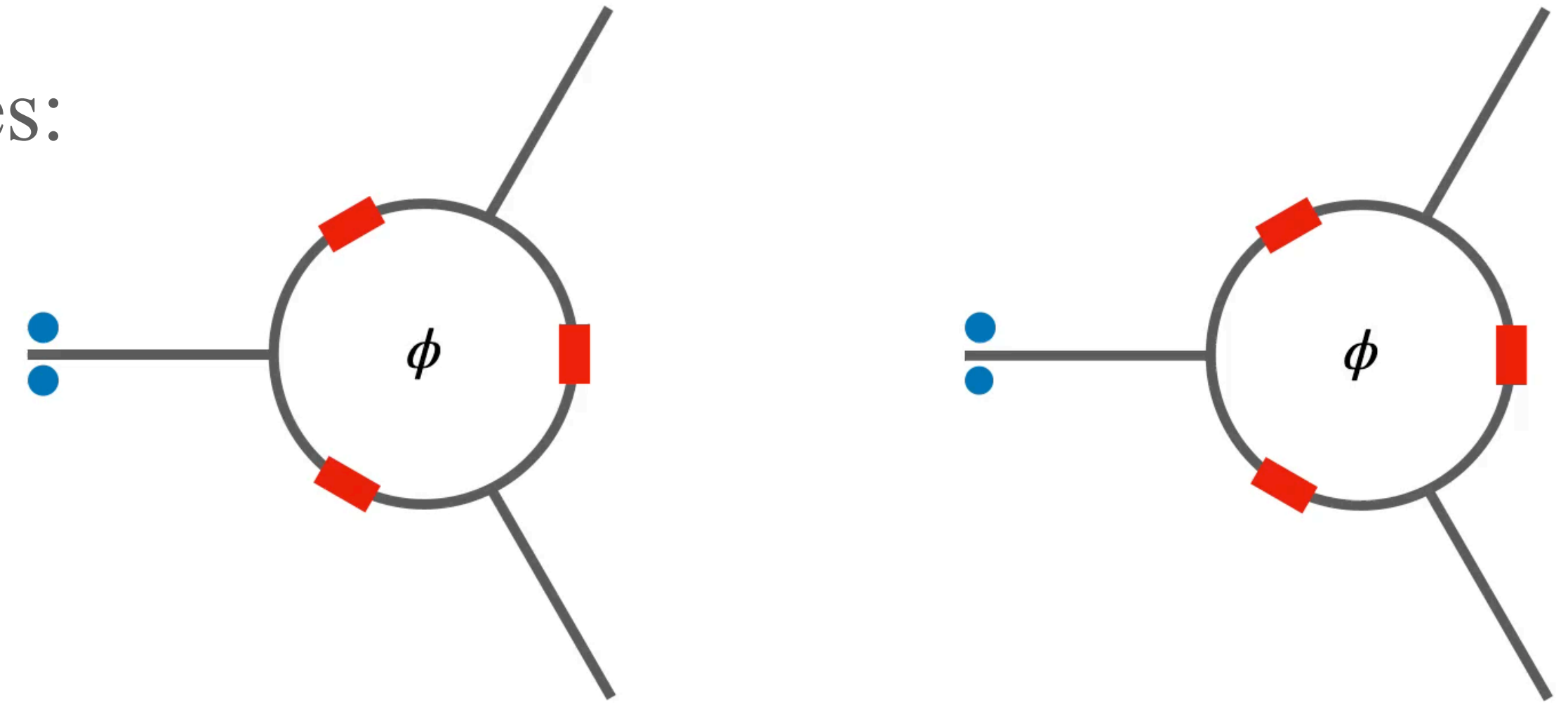
# Transport processes

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- There are two competing transport processes:

⇒ Charge carrier is a **Cooper pair**

⇒ Charge carrier is a **single electron**



- Consider magnetic flux  $\phi = \frac{1}{2}(2n + 1)\pi$ , (in units of flux quanta  $\Phi_0 = h/2e$ )
- The Aharonov Bohm phase  $2\phi$  is **destructive for the pair**, but **non-destructive for single electrons**

# Large coherence length

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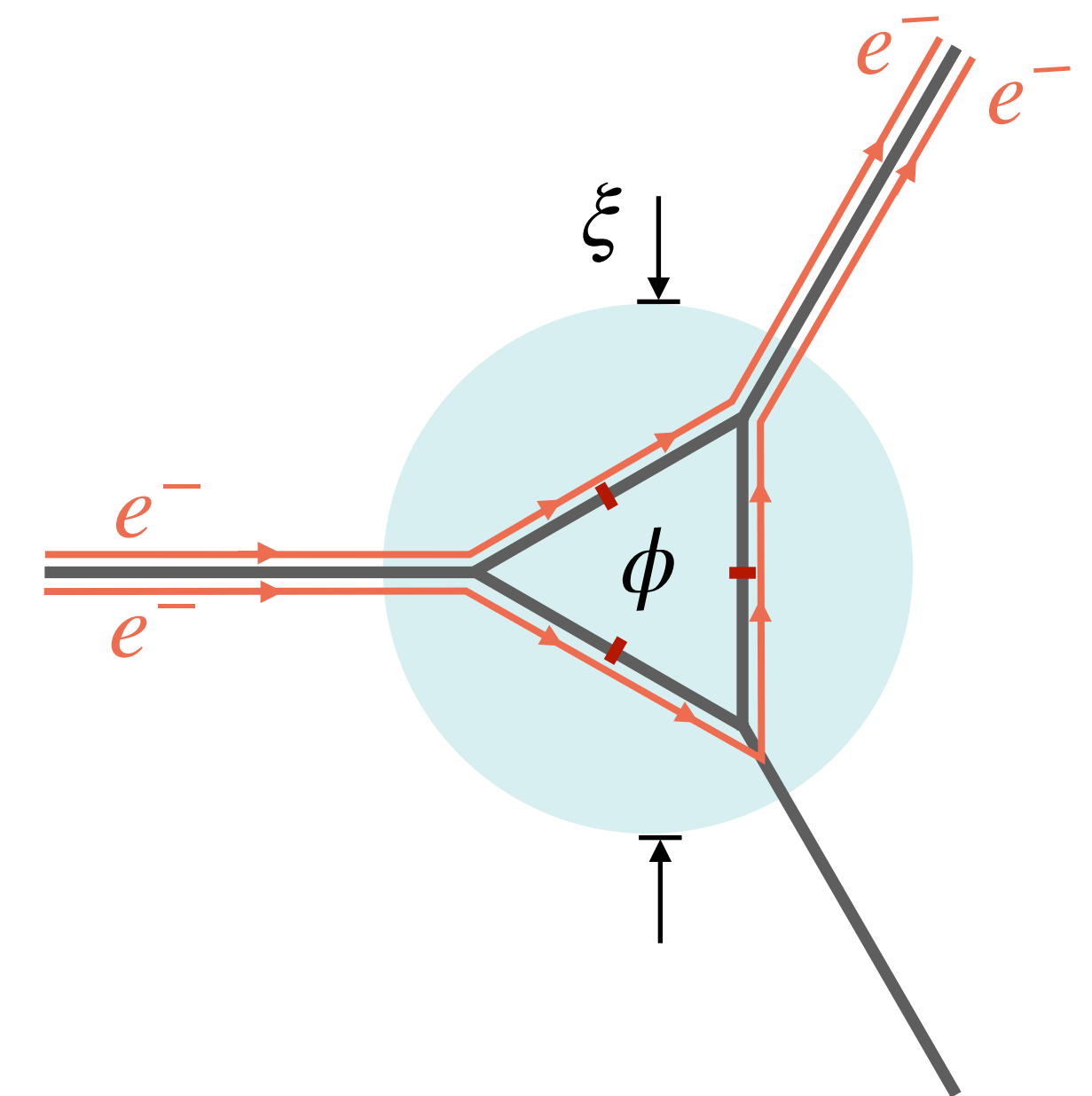
- The superconducting gap  $\Delta$  induces a coherence length,

$$\xi \sim \frac{1}{\Delta}$$

This is a measure of how far away electrons in a Cooper-pair can separate from each other

⇒ Large superconducting gap implies Cooper-pairs traveling tightly together: no splitting

⇒ We expect to see Cooper-pair splitting for **large enough**  $\xi$  (order of size of circulator)



# Numerics - Kwant

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- Describe the circulator in terms of an effective, mean-field, description:

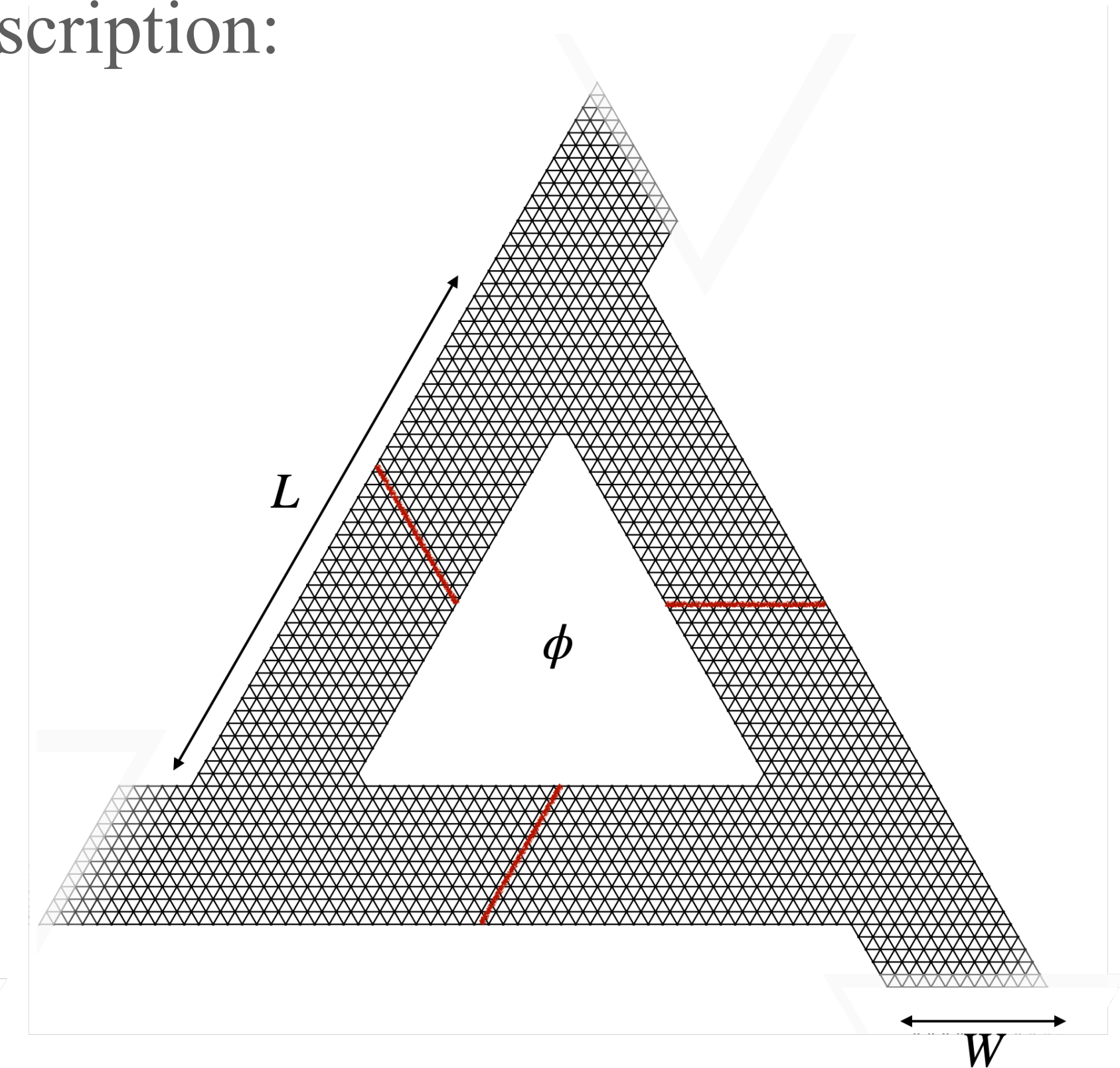
$$H = H_t + H_\Gamma + H_\Delta$$

where:

⇒  $H_t$  contains **intra-wire** electron hopping

⇒  $H_\Gamma$  contains **inter-wire** electron (Josephson weak link)

⇒  $H_\Delta$  contains **superconductivity** pairing terms



Numerical results through exact diagonalization and Kwant (Groth, Wimmer, Akhmerov, & Waintal, 2014)

# Physics of a circulator

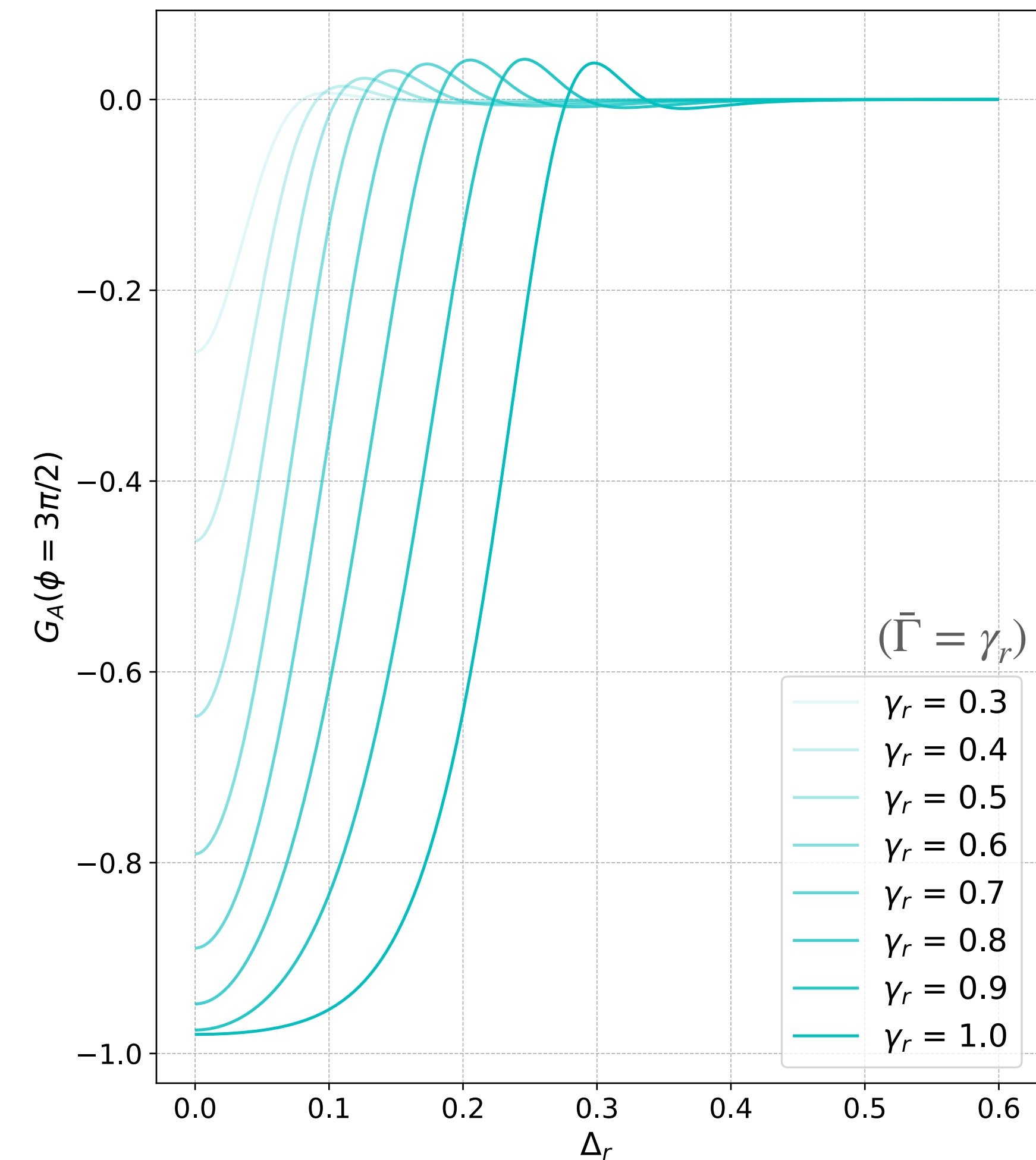
- Cooper-pair splitting can be captured by the **non-local antisymmetric conductance** (Wang & Hu, 2011)

$$G_A = \frac{1}{3} \epsilon_{ij} G_{ij}$$

- Let  $L$  the size of circulator and  $a$  lattice spacing. We define the dimensionless ratios  $\bar{\Gamma} = \Gamma/t$ ,  $\bar{\Delta} = \Delta/t$ ,  $\bar{L} = L/a$ .

- For semi-integer  $\phi = \frac{1}{2}(2n + 1)\pi$ , we find that

$$\frac{G_A(\bar{\Gamma}, \bar{\Delta}, \bar{L})}{e^2/h} = \frac{16\bar{\Gamma}^3}{(1 + 3\bar{\Gamma}^2)^2} f\left(\frac{\bar{\Delta}}{\bar{L}^\alpha \bar{\Gamma}^\beta}\right) \exp(-\bar{\Delta}\bar{L})$$

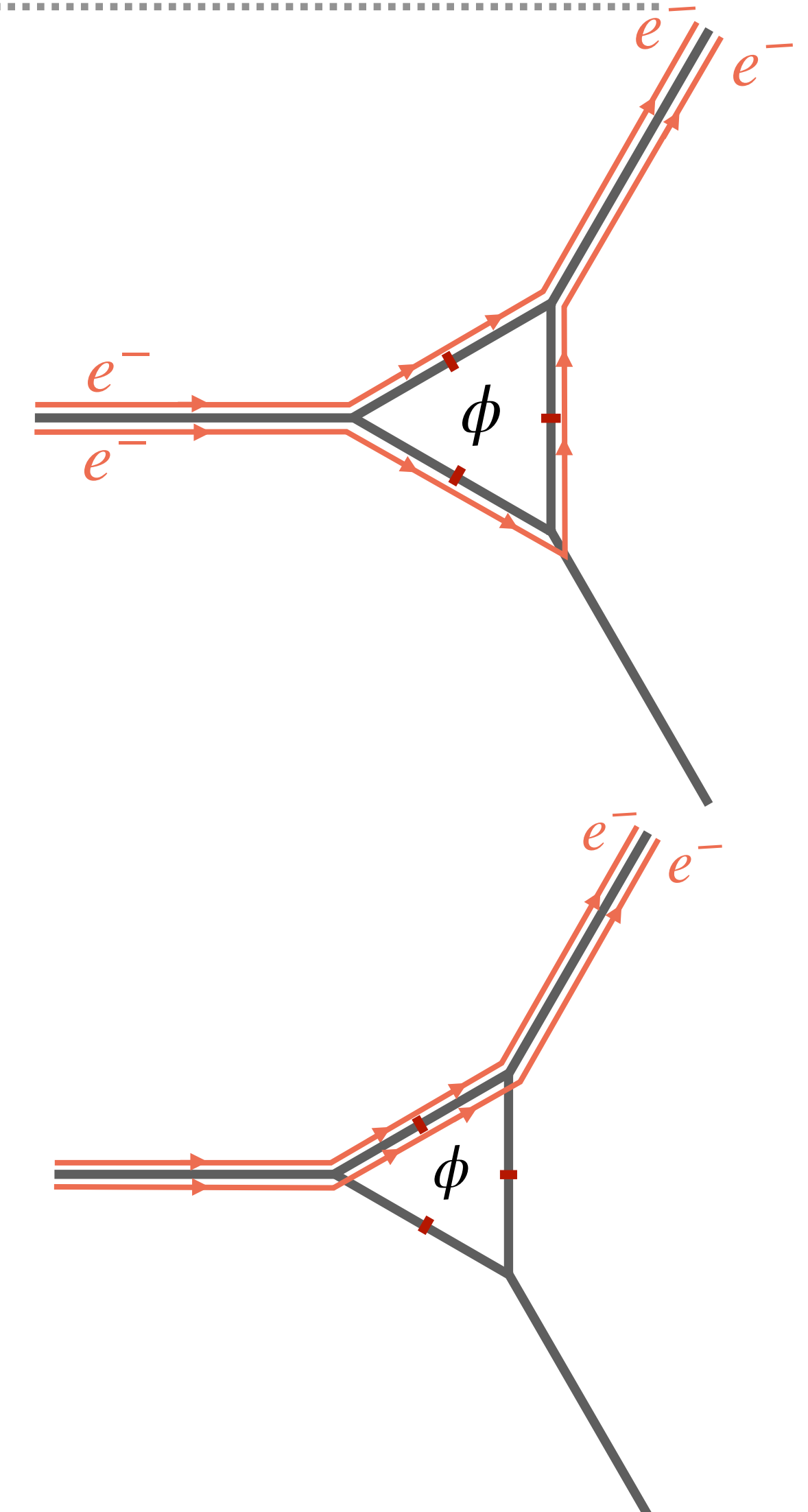


# Physics of a circulator

- The function  $f(x)$  increases to a peak at  $x \approx 2$  and then decays to zero

$$\frac{G_A(\bar{\Gamma}, \bar{\Delta}, \bar{L})}{e^2/h} = \frac{16\bar{\Gamma}^3}{(1 + 3\bar{\Gamma}^2)^2} f\left(\frac{\bar{\Delta}}{\bar{L}^\alpha \bar{\Gamma}^\beta}\right) \exp(-\bar{\Delta}\bar{L})$$

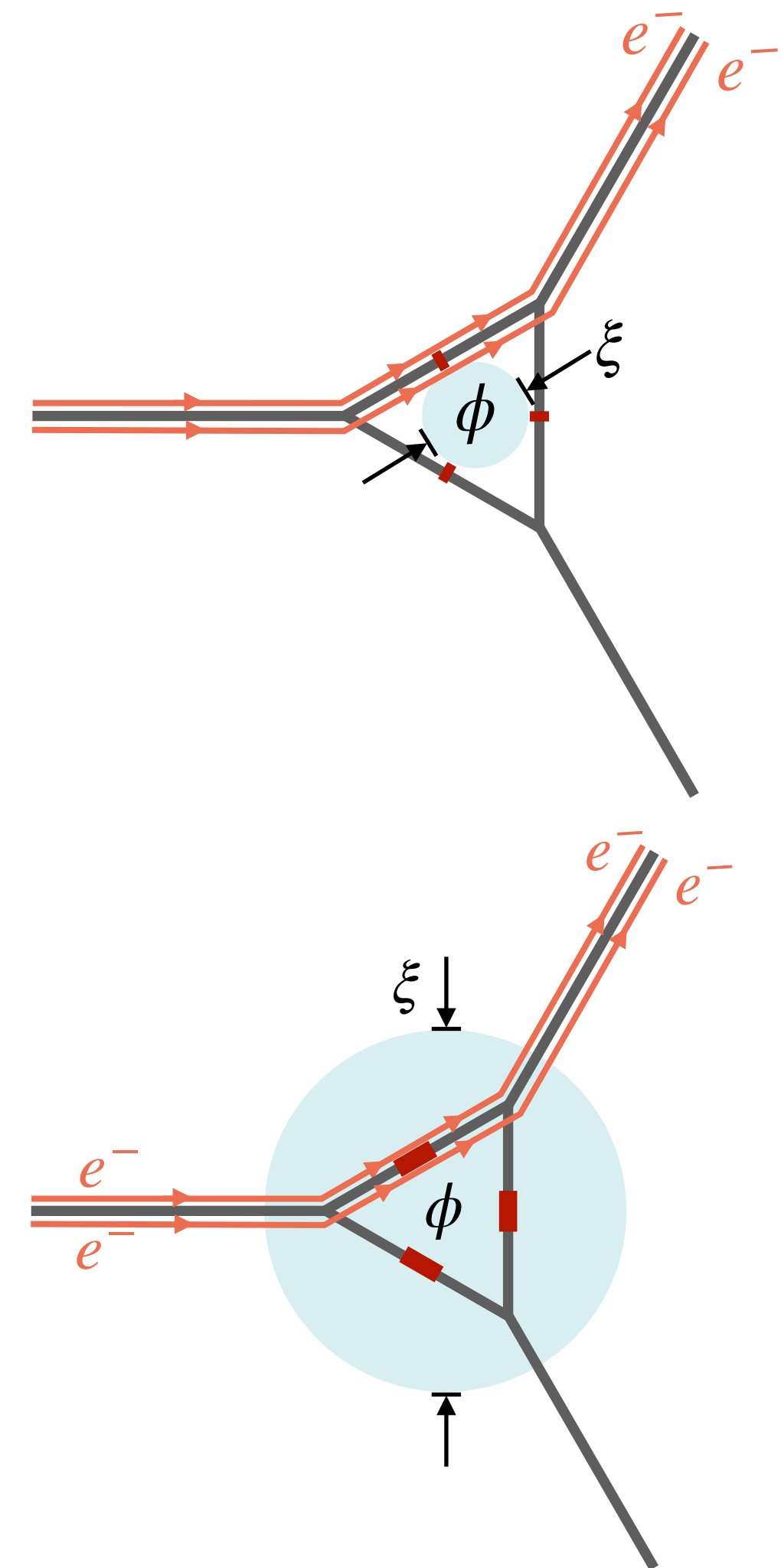
- Cubic scaling with  $\Gamma$  suggests a third order process, instead of order two
- The exponential dependence introduces a coherence length  $\xi \sim \Delta^{-1}$



# Splitting regimes

$$\frac{G_A(\bar{\Gamma}, \bar{\Delta}, \bar{L})}{e^2/h} = \frac{16 \bar{\Gamma}^3}{(1 + 3\bar{\Gamma}^2)^2} f\left(\frac{\bar{\Delta}}{\bar{L}^\alpha \bar{\Gamma}^\beta}\right) \exp(-\bar{\Delta} \bar{L}),$$

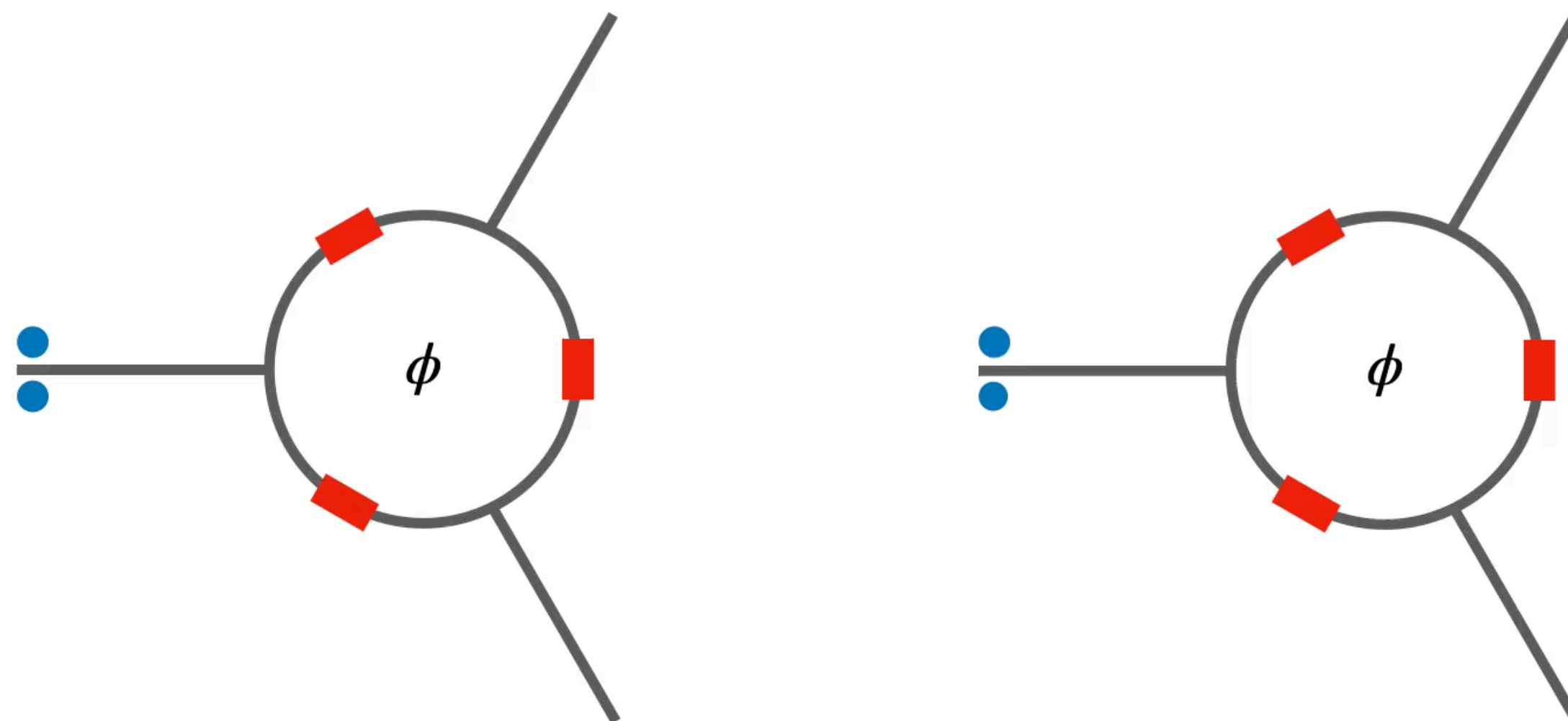
- For **small coherence** length,  $\bar{L} \gg \bar{\Delta}^{-1}$ , the exponential decay dominates, causing  $G_A \rightarrow 0$ , reflecting the absence of pair splitting.
- For **barriers that are too thick**,  $\bar{\Gamma}^\beta \ll \bar{\Delta} \bar{L}^\alpha$ , the argument  $x \gg 2$  of the function  $f(x)$  becomes too large, placing the system outside the relevant pair-splitting regime.



# Takeaway

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- In order to observe **splitting**, we need three key ingredients:
  - Semi integer multiple of  $\pi$  magnetic flux  $\phi = \frac{1}{2}(2n + 1)\pi$
  - Large coherence length  $\xi \gg 1$
  - Barriers in Josephson junctions  $\Gamma$  cannot be too thick

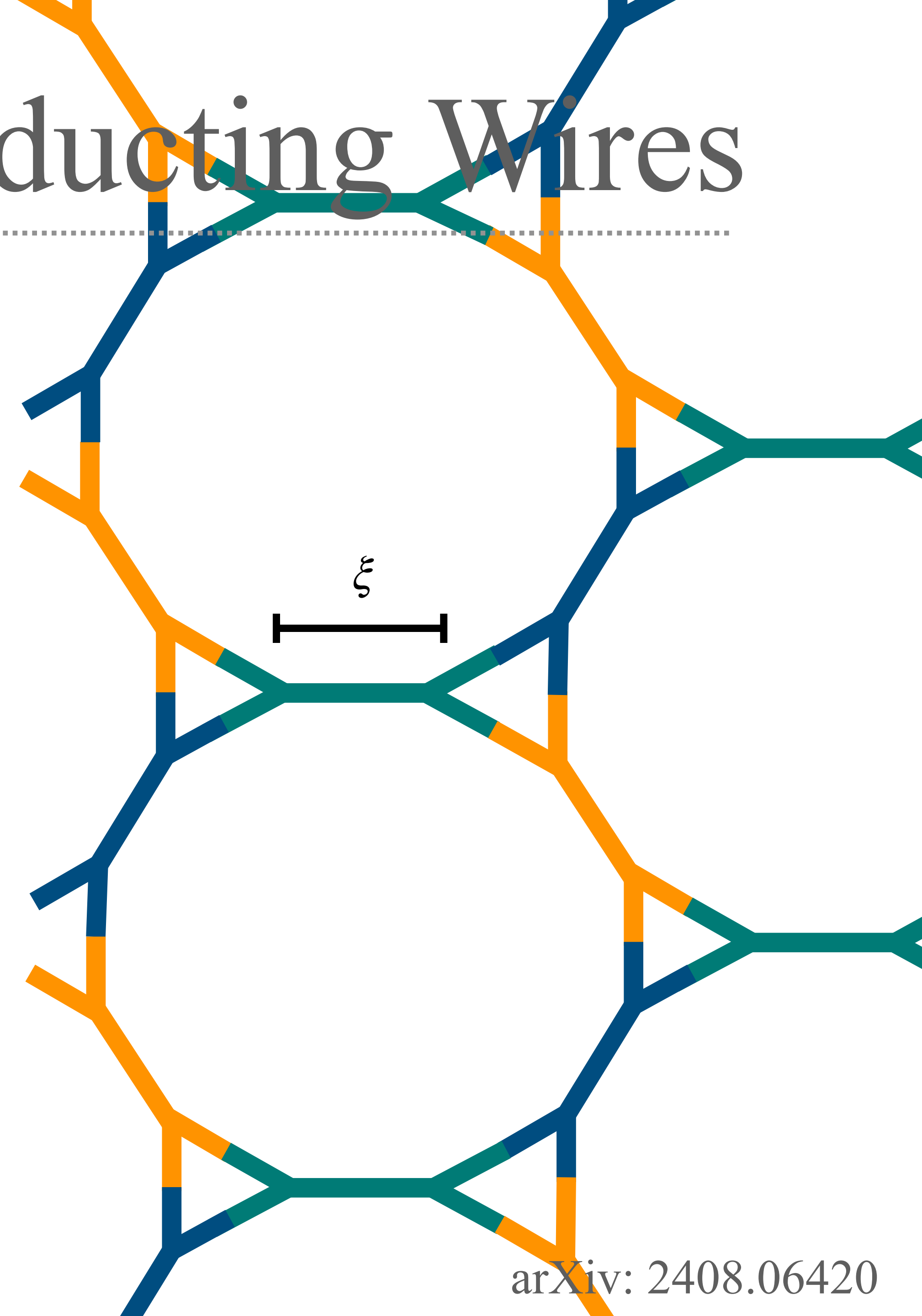


# Networks of Superconducting Wires

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- For large coherence length, each wire behaves as a single degree of freedom
- **Meta-material:** array of mesoscopic wires
- Insert appropriate flux through the network
- Hope to see some sort of Chern insulator physics, but where the underlying dof are superconductors

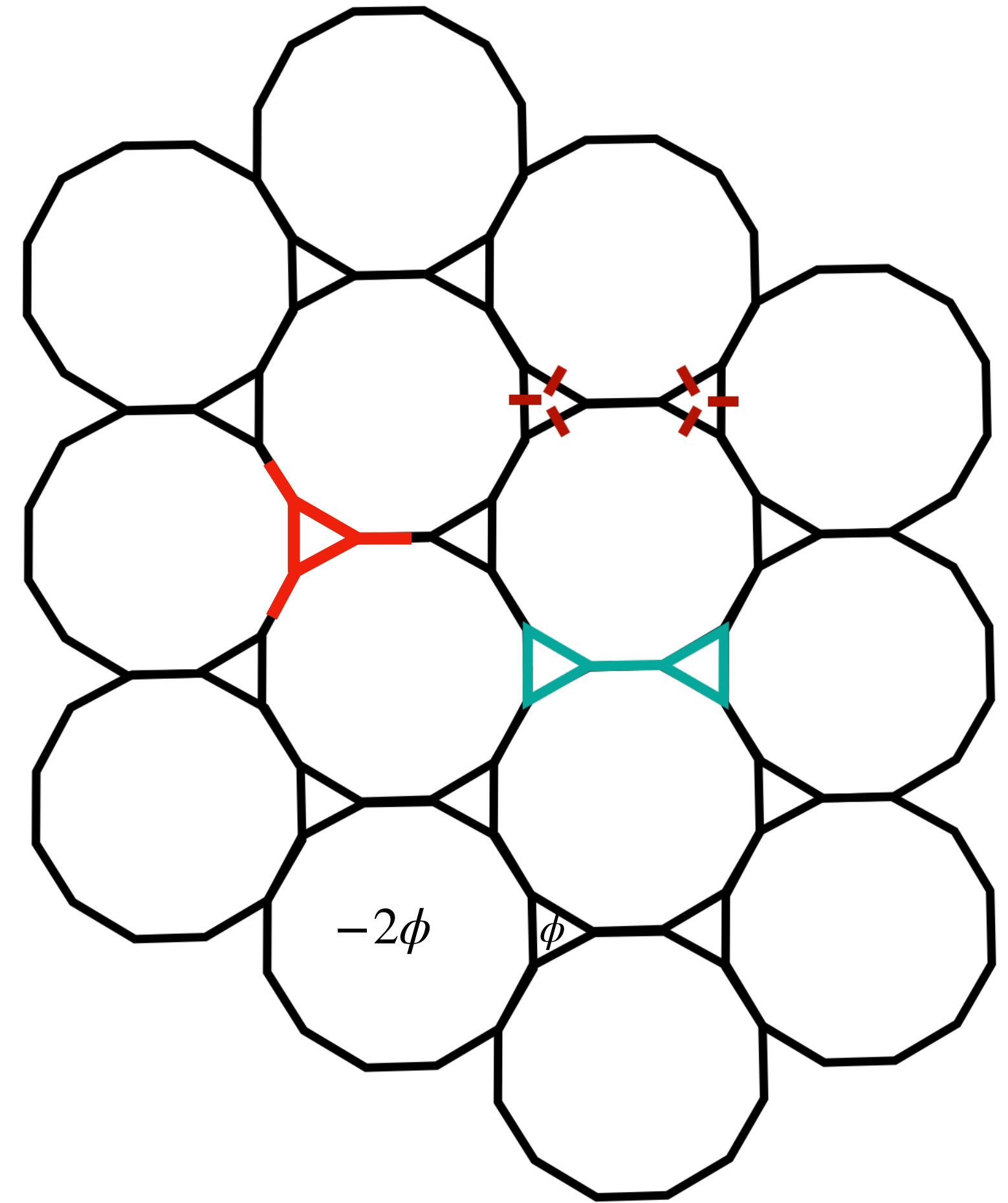
⇒ **Topological superconductors** (?)



# Network of circulators

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- Network: Connect leads of Y-splitters to each other
- Archimedean  $(3,12^2)$  geometry: triangles connected to dodecagons
- Flux of  $\phi$  through triangles and  $-2\phi$  through the dodecagons, which is only defined mod  $2\pi$
- Different arrangements can, in principle, lead to different behaviors of the system



# Effective Model

- Describe the network of wires in terms of an effective model:
- Large coherence length: approximate each wire as a single site

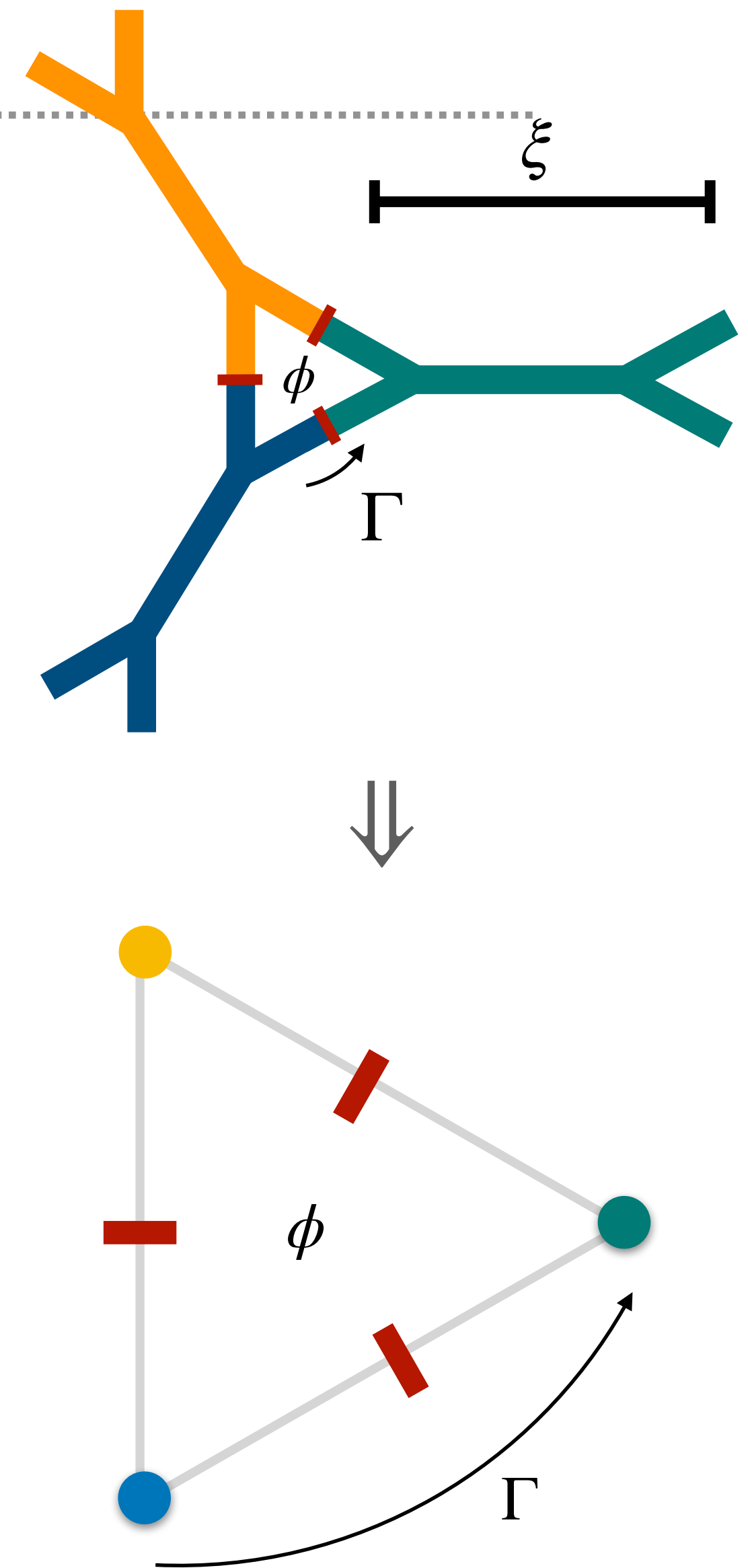
$$H = H_{\Gamma} + H_{\Delta}$$

where:

$\Rightarrow H_{\Gamma}$  contains electron hopping between neighboring sites

$\Rightarrow H_{\Delta}$  contains superconductivity pairing terms

Intra-wire hopping terms  $H_t$  renormalize the effective parameters



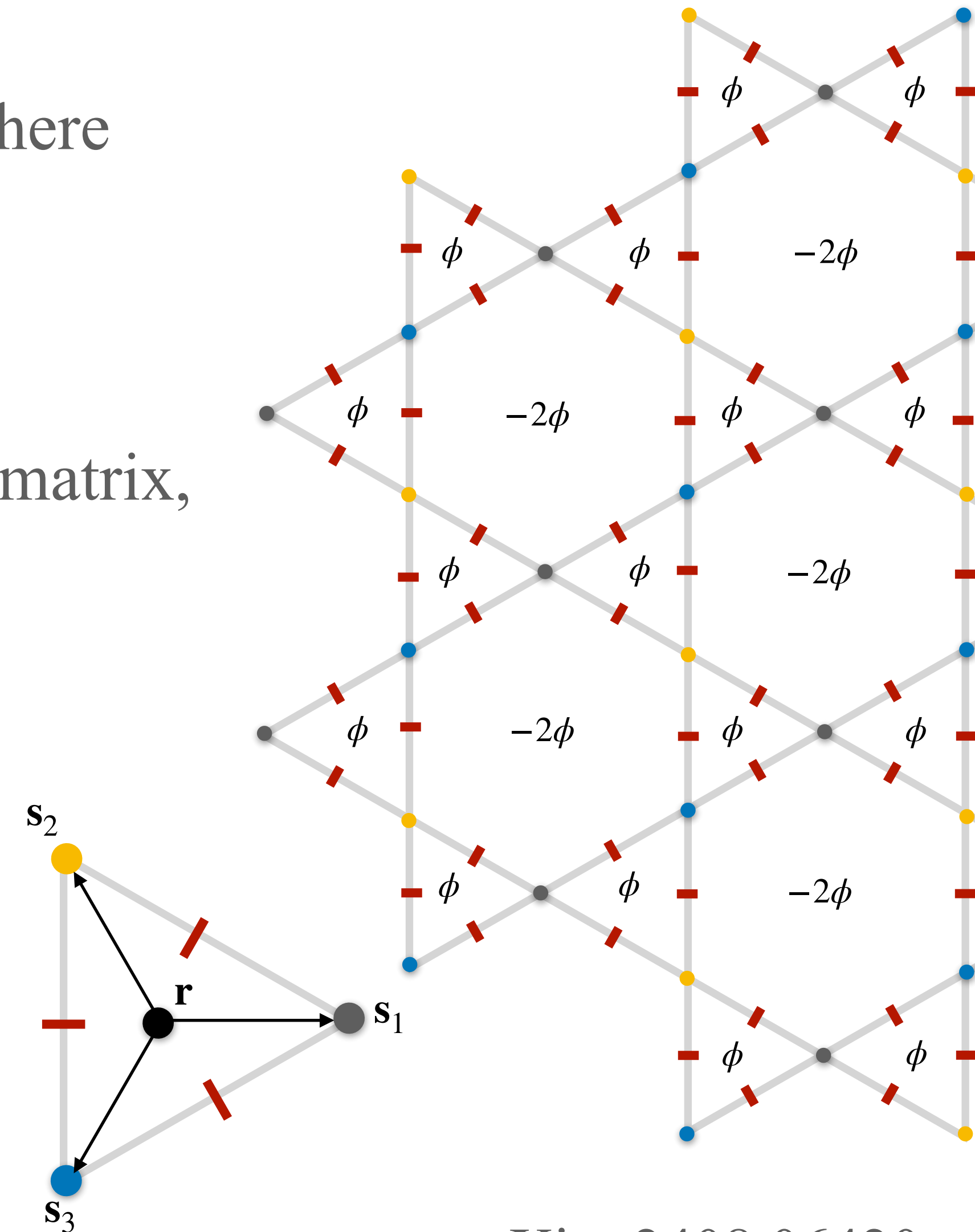
# Ugly Hamiltonian

- The mean-field matrix Hamiltonian is  $H_k = \begin{bmatrix} h_k & \Delta_k \\ \Delta_k^* & -h_{-k}^* \end{bmatrix}$ , where

$$h_k = \begin{bmatrix} 0 & \alpha_{\triangleright} + \alpha_{\triangleleft} d_k^{12} & \bar{\alpha}_{\triangleright} + \bar{\alpha}_{\triangleleft} d_k^{13} \\ \bar{\alpha}_{\triangleright} + \bar{\alpha}_{\triangleleft} d_k^{21} & 0 & \alpha_{\triangleright} + \alpha_{\triangleleft} d_k^{23} \\ \alpha_{\triangleright} + \alpha_{\triangleleft} d_k^{31} & \bar{\alpha}_{\triangleright} + \bar{\alpha}_{\triangleleft} d_k^{32} & 0 \end{bmatrix} \text{ is the hopping matrix,}$$

$$\Delta_k = \Delta \begin{bmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{bmatrix} \text{ is the superconducting matrix}$$

In the above,  $d_k^{ij} = e^{-ik \cdot (s_i - s_j)}$  and  $\alpha_{\triangleright} = \alpha_{\triangleleft} \equiv \Gamma e^{i\phi/3}$



# Phase Diagram

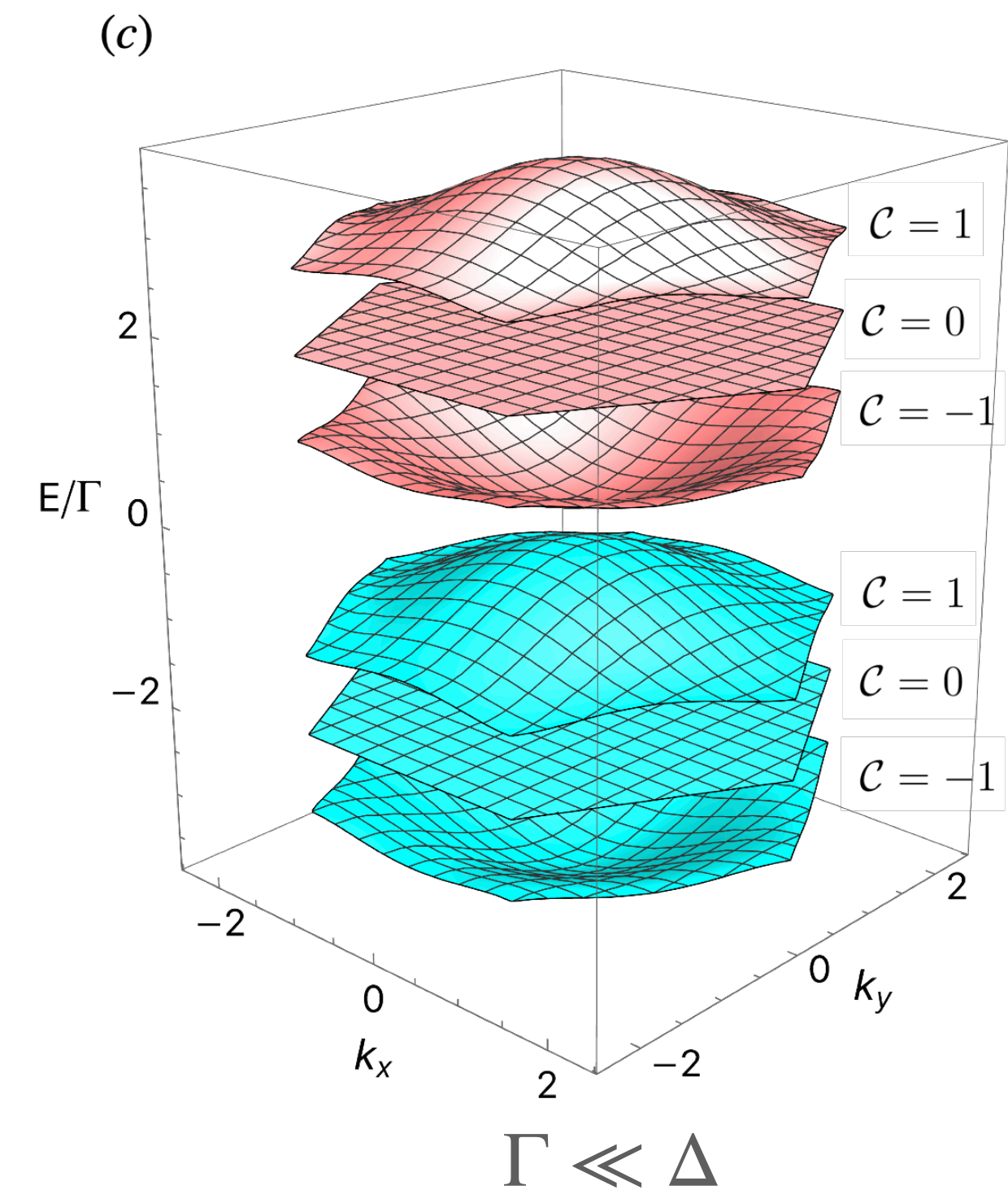
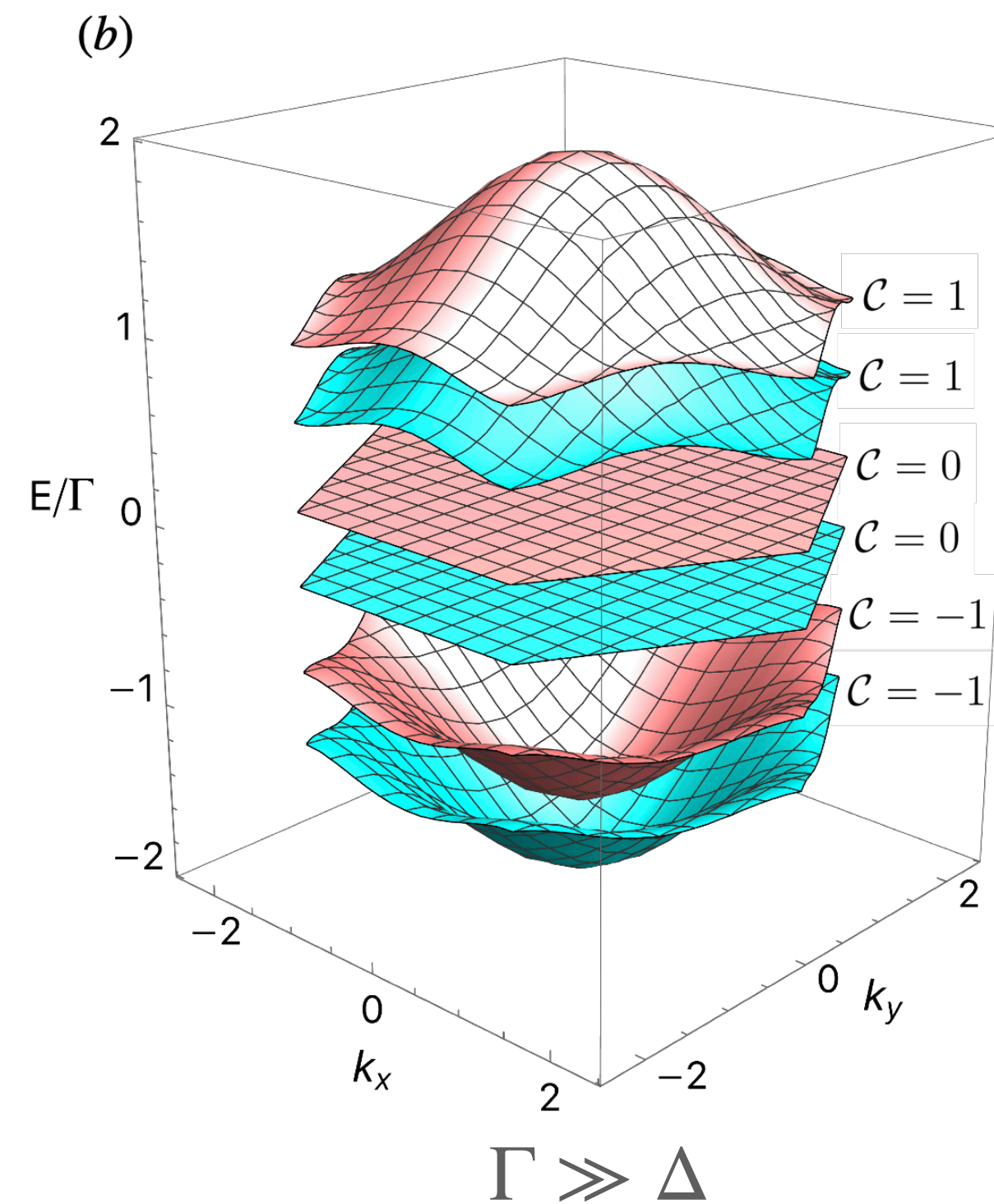
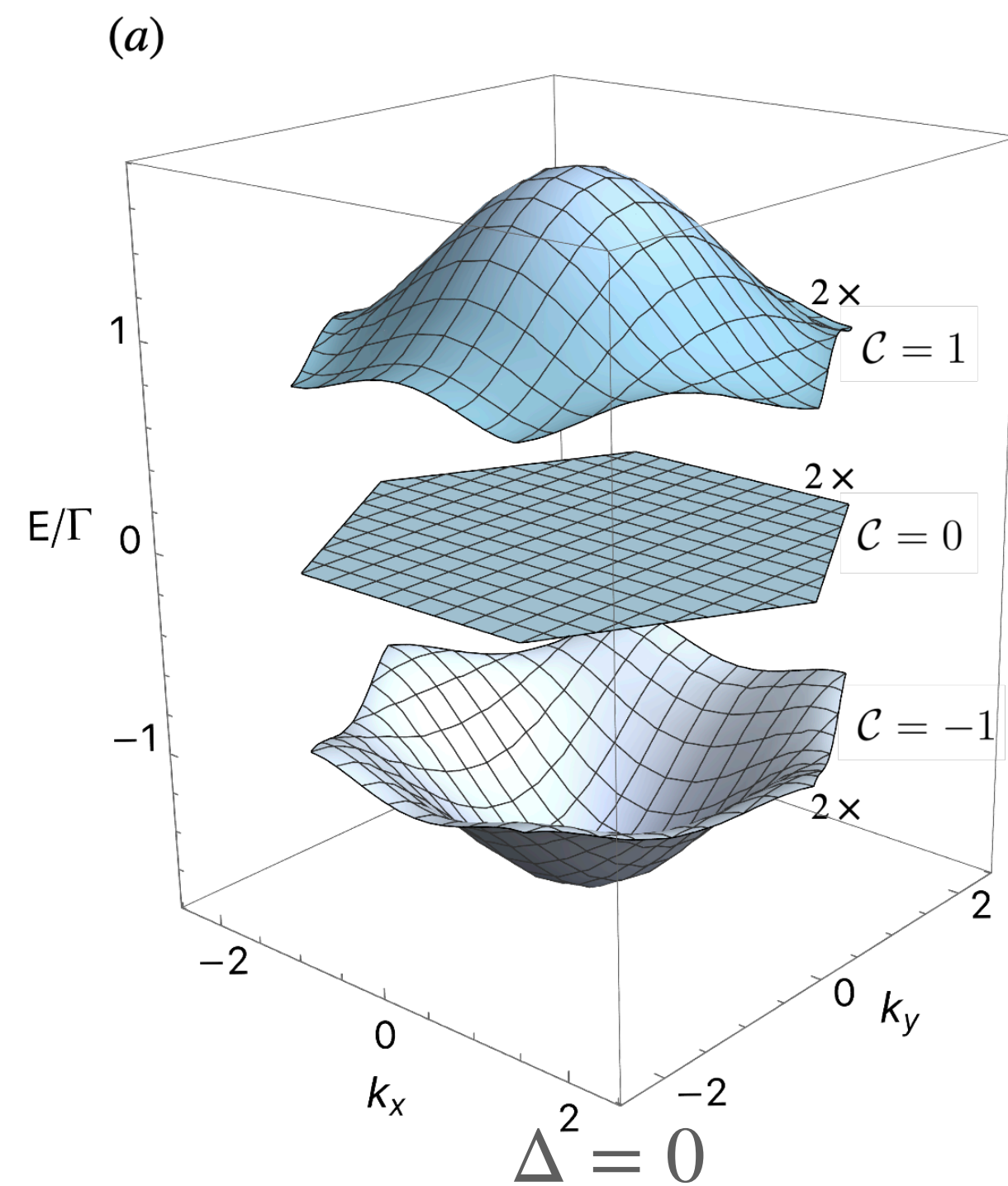
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- Express the phase diagram in terms of  $\Gamma/\Delta$  and  $\phi$ . For this, we :
  - $\Rightarrow$  Compute the many-body **ground state energy** as a function of  $\theta_a$  parameters
  - $\Rightarrow$  Solve for the configurations  $\theta_a^*$  **that minimize** the total energy
  - $\Rightarrow$  Compute **Chern numbers** in momentum space
- Define chirality,  $\chi \sim \sin(\Delta\theta)$  which indicates JJ current direction

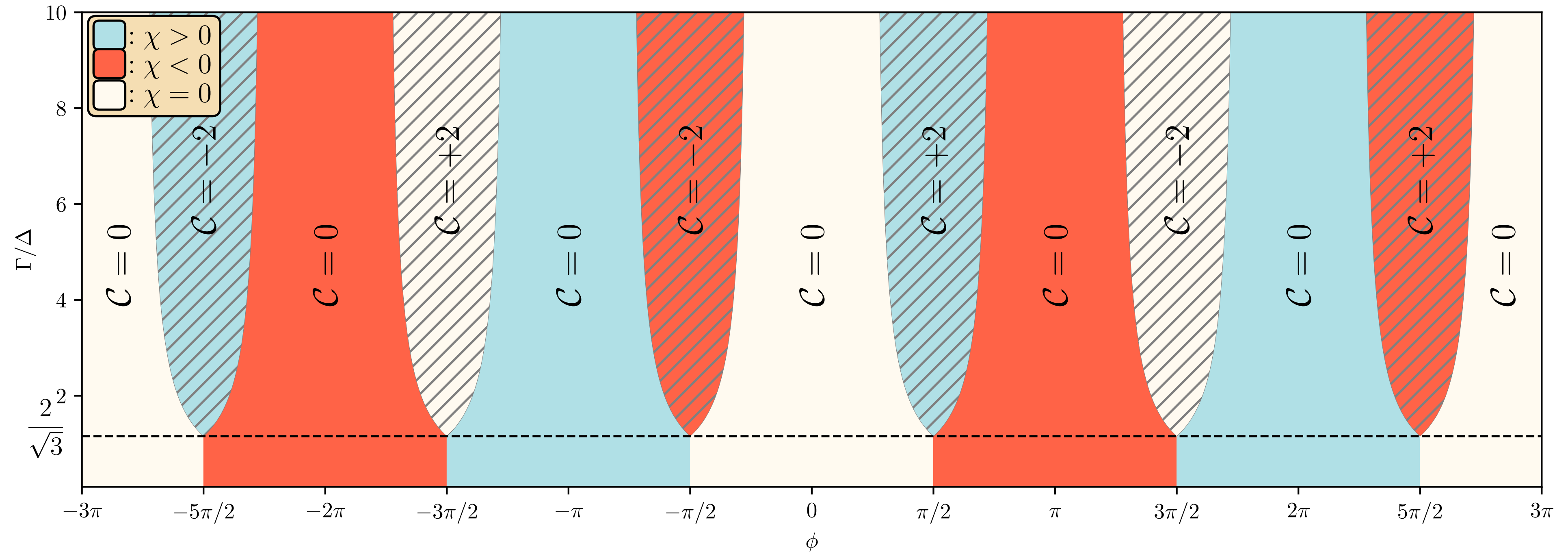
# Exactly solvable case: $\phi = 3\pi/2$

- For  $\phi = 3\pi/2$ , numerically find that all  $\theta_a = 0$ . Then, energy bands are given by.  $\pm\Delta$ ,  $-\epsilon_{\mathbf{k}} \pm \Delta$ ,  $\epsilon_{\mathbf{k}} \pm \Delta$ , with

$$\epsilon_{\mathbf{k}} = \frac{\Gamma}{\sqrt{2}} \sqrt{3 + \cos(\sqrt{3}k_x) + \cos\left(\frac{\sqrt{3}}{2}k_x - \frac{3}{2}k_y\right) + \cos\left(\frac{\sqrt{3}}{2}k_x + \frac{3}{2}k_y\right)}.$$



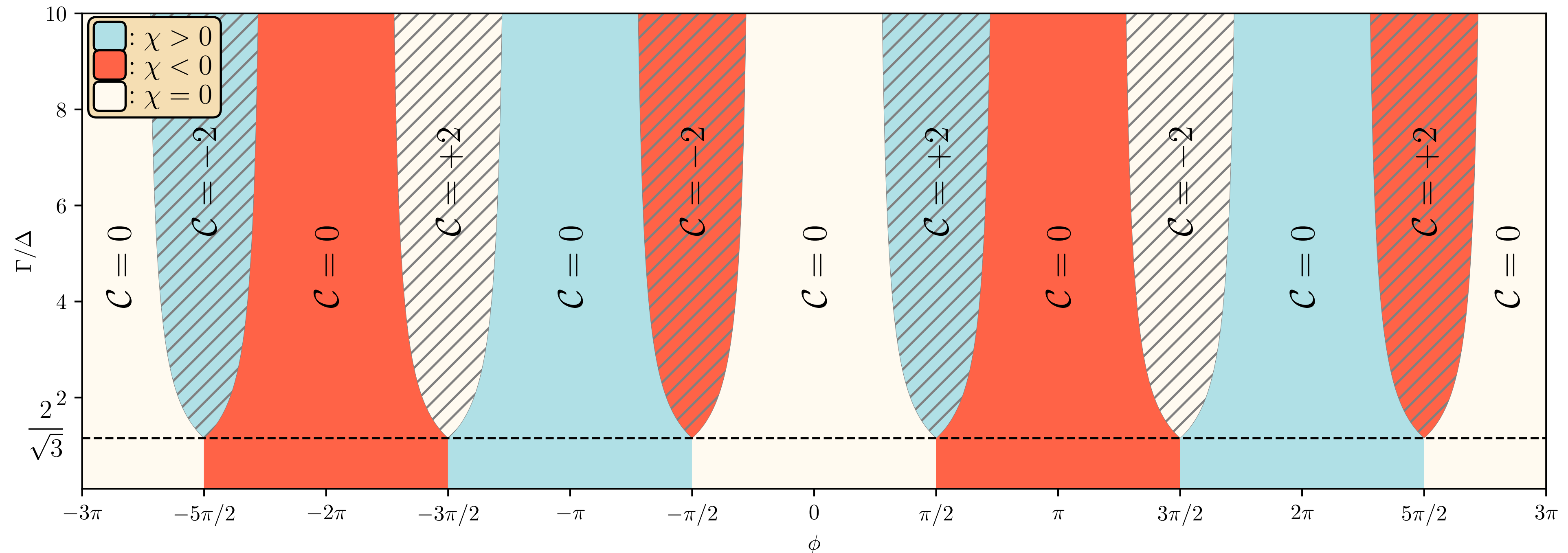
# Phase Diagram



- Alternating stripes of trivial ( $C = 0$ ) and **topological superconducting phases** ( $C \neq 0$ )
- **Critical value**  $(\Gamma/\Delta)^* = 2/\sqrt{3}$ : below such a value, all phases are topologically trivial
- Below the critical line  $(\Gamma/\Delta)^*$ , chirality exactly match classical configurations that minimize

$$H_{JJ} = -E_J \left[ \cos\left(\theta_2 - \theta_1 + \frac{2\phi}{3}\right) + \cos\left(\theta_1 - \theta_3 + \frac{2\phi}{3}\right) + \cos\left(\theta_3 - \theta_2 + \frac{2\phi}{3}\right) \right].$$

# Phase Diagram

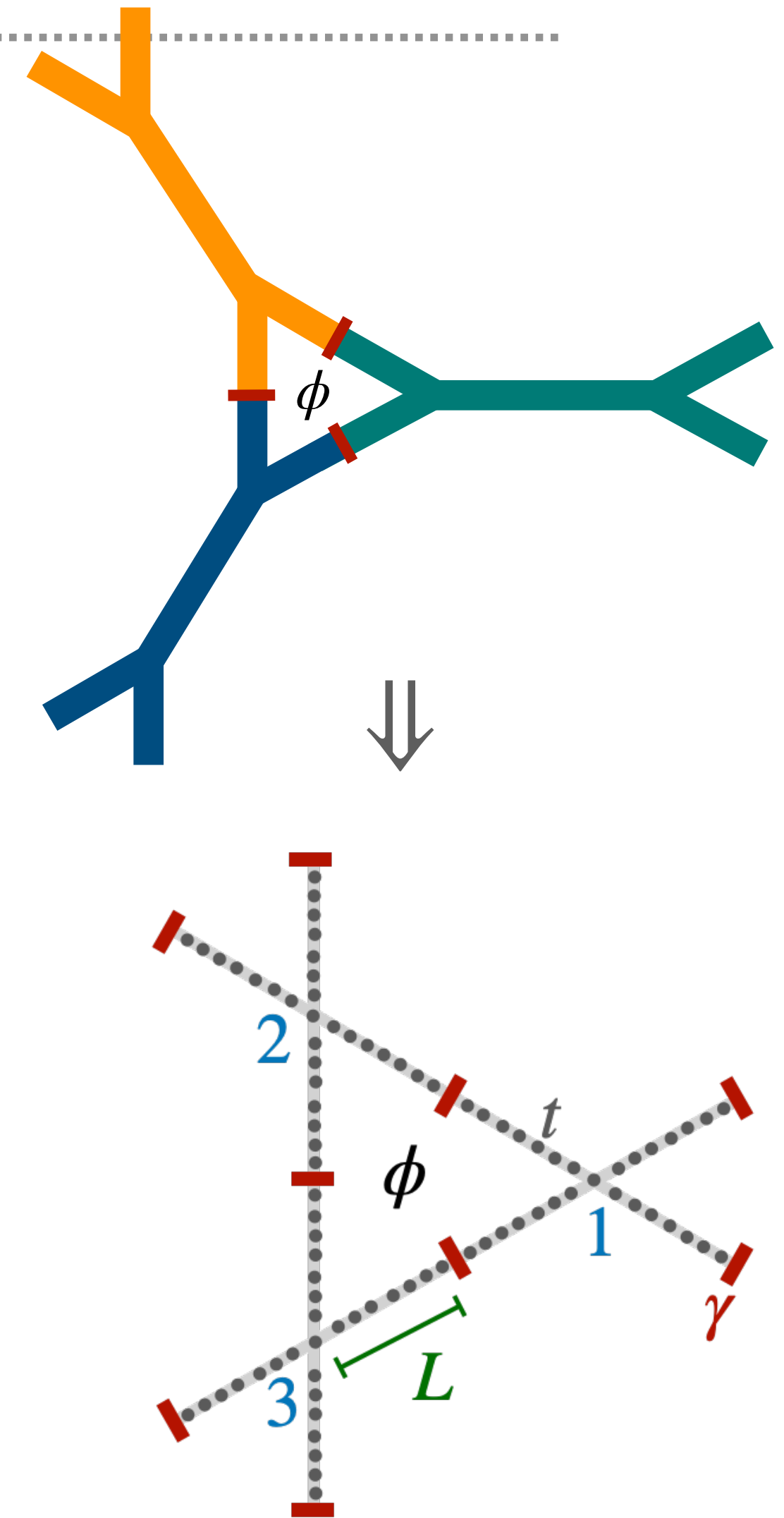


- For flux integer multiples of  $\pi$  time-reversal symmetry is restored: vanishing Chern-numbers
- Topological superconductors phases appear at **maximally breaking of time-reversal**: half-integer flux values  $\Rightarrow$  transport properties are dominated by **Cooper-pair splitting**
- The consecutive splitting of Cooper-pairs at each node builds up to robust topological properties

# Microscopic model

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- Later in the paper we **justify** our previous approximation
  - ⇒ We consider each extensive wire as a set of  $4L$  dots
- We analytically find all the orbital wave functions and numerically compute the energy bands and Chern numbers
  - ⇒ The **results agree with our effective description**, a more intuitive picture



# Final Remarks

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- ⇒ Cooper-pair splitting require a balance between **superconductivity, flux and JJ junctions**
- ⇒ Observe a direct relation between **pair splitting** and **topological superconductivity** in a network of circulators

## Future Perspectives:

- Explore more general networks of hybrid circuit elements
- Design richer (non-Abelian, fracton, etc) topological phases (?)

Thank you!

[arXiv: 2408.06420](https://arxiv.org/abs/2408.06420)



# Parameters

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- Aluminum  $\xi \sim 1.6\mu m$
- Typical sizes of wires:  $\sim 500$  nm
  - Arrays with 1000 x 1000 (Nb) wires (Sohn et al, 1993)
- Temperature  $T \sim 1.2K$
- Critical magnetic field  $H_c \sim 0.01T$   
(enough to get us desired  $\phi$  values)