U(1) Enriched Topological Order

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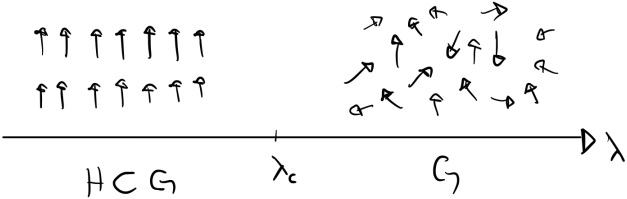
Classical Phases of Matter

• **Classical phases** of matter and phase transitions are described by spontaneous symmetry breaking of symmetry groups G down to $H \subset G$

$\sim$ Local order parameter

$\sim$ Transition between symmetric and broken phases

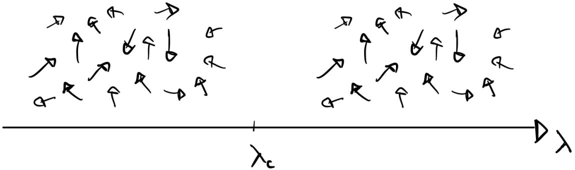


$\sim$ Effective description: Landau-Ginsburg theory

Topological Phases of Matter

• Topological order correspond to (gapped) **quantum phases** of matter

∼ No local order parameter



∼ Long range entanglement;

∼ States cannot be distinguished locally

∼ Low-energy descriptions in terms of Topological Quantum Field Theories (TQFTs)

Quantum Spin Liquids

- (Gapped) Quantum Spin Liquids (QSLs) are examples of topologically ordered phases of matter

- Does not present magnetic order (no local order parameter)

- Very robust against arbitrary perturbations

- In this work we deal with spin 1/2 systems in two spatial dimensions

Pauli Matrices 101

Pauli matrices $\sigma^x, \sigma^y, \sigma^z$ act on spin 1/2 degrees of freedom, with a Hilbert space of dimension 2. They are unitary, Hermitian 2×2 matrices that obey

$$(\sigma^x)^2 = (\sigma^y)^2 = (\sigma^z)^2 = 1 \quad \text{and} \quad \{\sigma^k, \sigma^{k'}\} = 0 \tag{1}$$

for $k \neq k'$.

We choose the basis states to diagonalize σ^z, which we call $|\uparrow\rangle, |\downarrow\rangle$

$$\sigma^z |\uparrow\rangle = +1 |\uparrow\rangle , \quad \sigma^z |\downarrow\rangle = - |\downarrow\rangle \tag{2}$$

The action of σ^x pauli matrix on these states is to flip one into the other

$$\sigma^x |\uparrow\rangle = |\downarrow\rangle , \quad \sigma^x |\downarrow\rangle = |\uparrow\rangle . \tag{3}$$

Many Body System

Tensor Hilbert space of dimension 2^N, where N is the total number of spins.
Basis

$$|\uparrow,\uparrow,\ldots,\uparrow\rangle\,, \quad |\downarrow,\uparrow,\ldots,\uparrow\rangle\,,\ldots \tag{4}$$

Many spins σ_ℓ^k which are bosonic

$$[\sigma_\ell^k,\sigma_{\ell'}^{k'}]=0, \quad \text{if} \quad \ell\neq\ell'. \tag{5}$$

Action of σ_ℓ^x at position ℓ is to flip spin at that position

$$\sigma_1^x\,|\uparrow,\uparrow,\ldots,\uparrow\rangle=|\downarrow,\uparrow,\ldots,\uparrow\rangle\,. \tag{6}$$

Simplest Example: Toric Code Hamiltonian

- Proposed by Kitaev in 2003 (Kitaev, Alexey. Annals of physics 303.1 (2003): 2-30.), the model is defined as follow

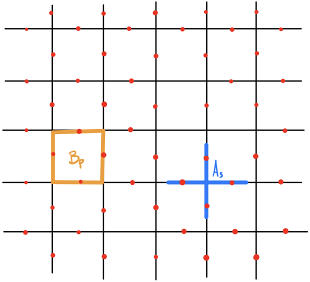
A spin 1/2 in each link of a two-dimensional square lattice

$\sim$ The Hamiltonian is given by

$$H = -\sum_s A_s - \sum_p B_p \tag{7}$$

for star and plaquette operators

$$A_s = \prod_{\ell \in s} \sigma_\ell^x, \quad B_p = \prod_{\ell \in \partial p} \sigma_\ell^z. \tag{8}$$



All operators commute with each other

$$[A_s, A_{s'}] = 0, \quad [B_p, B_{p'}] = 0, \quad [A_s, B_p] = 0. \tag{9}$$

Properties:

• Studying the properties of the system sums up to solving the time independent Schroedinger's equation

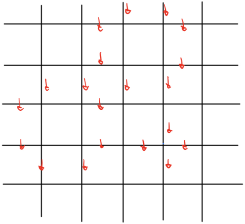
$$H\left|\psi\right\rangle = E\left|\psi\right\rangle . \tag{10}$$

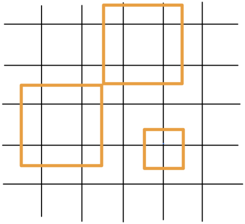
The ground state $|GS\rangle$ minimizes the energy, obeying

$$A_s\left|GS\right\rangle = +1\left|GS\right\rangle \quad \text{and} \quad B_p\left|GS\right\rangle = +1\left|GS\right\rangle \tag{11}$$

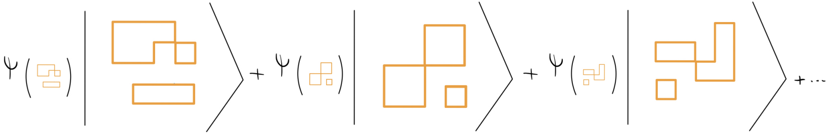
for all s and p in the lattice

$\sim$ Introduce a picture of strings in the σ^z basis

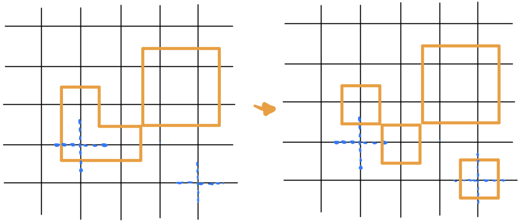




Then, spin configurations that have all $B_p = +1$ correspond to arbitrary closed loop configurations

$|\psi\rangle =$ 

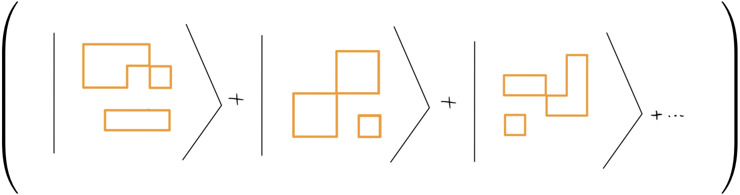
• The condition that the ground state has eigenvalue $+1$ for all A_s impose constraints over the amplitudes $\psi(C)$



$\Rightarrow$ One can **create**, **anihilate**, and **deform strings**

It is not hard to convince one-self that the ground state is given by the equal weighted superposition $\psi(C) = \psi(C')$ for all C'.

$$|GS\rangle = \frac{1}{\sqrt{N}}$$



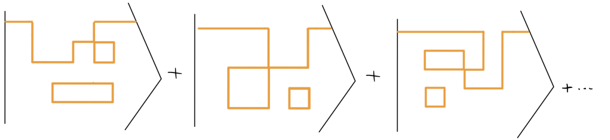
• One can explicitly compute expectation values in the theory, as $\langle GS|\, \sigma^z_\ell \,|GS\rangle$. One can show that for any local operator

$$\langle GS|\, \mathcal{O} \,|GS\rangle = 0, \tag{12}$$

where $\mathcal{O}$ is build out of products of the Pauli operators in a local region $\Rightarrow$ **No local order parameter!**

The ground state, however, is not unique in lattices with non-trivial topology. For periodic boundary conditions, the following state is also a ground state

$$T_x |GS\rangle = \frac{1}{\sqrt{N}} \left(\; \cdots \; + \; \cdots \; + \; \cdots \; + \cdots \right),$$



where $T = \prod_\gamma \sigma^x_\ell$.

- In general, ground state degeneracy for closed lattices of genus g

$$GSD \;=\; 2^{2g}. \tag{13}$$

$\sim$ This is the historical motivation for the name *topological order*.

- The non-local string operators $W(\gamma) = \prod \sigma^z$ work as "order parameters" to distinguish the ground states $\rightarrow$ **Long range entanglement!**

On a torus ($g = 1$), there are four states, that we label

$$
\begin{aligned}
|{+}{+}\rangle &\equiv |GS\rangle\,, & |{+}{-}\rangle &= T_x\,|{+}{+}\rangle\,, \\
|{-}{+}\rangle &= T_y\,|{+}{+}\rangle\,, & |{-}{-}\rangle &= T_x T_y\,|{+}{+}\rangle\,,
\end{aligned}
\tag{14}
$$

with ± 1 the quantum numbers of W_x and $W_y \Rightarrow |W_x W_y\rangle$

• Long range entanglement is the responsible for the robust properties of the ground state. One cannot lift the degeneracy by local perturbations

$\Rightarrow$ **Ideal for storing quantum information!**

• Excitations correspond to exotic particles, called **anyons**. These present fractional statistics and obey fusion rules

• Toric code realizes a $\mathbb{Z}_2$ **Quantum Spin Liquid**

• We study QSLs for over 50 years since it was proposed by Anderson (Anderson, Philip W. Materials Research Bulletin 8.2 (1973): 153-160.) to describe High-Tc superconductivity. But only recently people managed to realize it experimentally Semeghini, Giulia, et al. Science 374.6572 (2021): 1242-1247.

• One needs more realistic model proposals, as WXY model which involves only two-body interactions (Chamon, C., Green, D., & Kerman, A. J. (2021). PRX Quantum, 2(3), 030341). The price to be paid is the addition of extra degrees of freedom.

• WXY can recover the toric code in a certain limit. It possesses a global $U(1)$ symmetry, which is the main motivation for our investigation (there is very little one can do in WXY model, both analytically or numerically)

$\Rightarrow$ **What happens when one impose a global $U(1)$ symmetry on top of topological order?**

Enrich Toric Code with $U(1)$ Symmetry

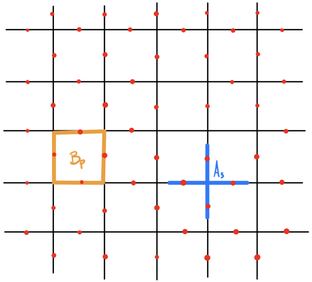
This model is believed to be an effective description for WXY model once one integrate out the extra degrees of freedom

$\sim$ The $U(1)$ symmetric Hamiltonian is defined by adapted star operators (Wu et al. arXiv: 2302.03707). Define:

$$A_s(\theta) = \prod_{\ell \in s} \left(\cos\theta\, \sigma_\ell^x + \sin\theta\, \sigma_\ell^y\right)$$

The $U(1)$ symmetric star is given by

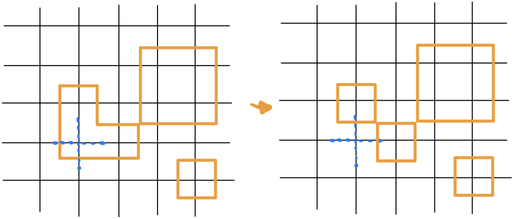
$$\mathcal{A}_s = \frac{1}{2\pi} \int_0^{2\pi} A_s(\theta)$$



Hamiltonian is given by:

$$H = -\sum_s \mathcal{A}_s - \sum_p B_p \quad (15)$$

• In the string picture introduced before, the new stars can no longer create loops, but only **deform** them, preserving the perimeter



This can be understood from the conserved quantity from Noether theorem:

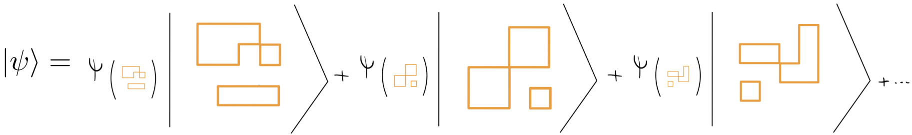
$$M = \sum_{\ell} \sigma_{\ell}^{z} \quad \Rightarrow \quad \text{Magnetization (perimeter of loops)} \tag{16}$$

$\sim$ The effect of the $U(1)$ symmetry is to preserve the total perimeter of the loops

Properties of the Model

• This model is no longer a sum of commuting terms and can no longer be solved exactly.

Thus, in the expansion,



one can no longer determine all the amplitudes $\psi(C)$ analytically.

$\Rightarrow$ We use Quantum Monte Carlo to study the model.

Six months latter:

- We find that on a torus, the ground state degeneracy is equals to three (naively, we would expect that $GSD = 4$).

- States are labelled by non-local operators

$$W_x = \prod_{\gamma_x} \sigma_\ell^z, \quad W_y = \prod_{\gamma_y} \sigma_\ell^z \tag{17}$$

We expect to have string-like operators T that connect the ground states to each other

$$|++\rangle, \quad |+-\rangle = T_x |++\rangle, \quad |-+\rangle = T_y |++\rangle \tag{18}$$

The state, $|--\rangle = T_x T_y |++\rangle$, however, is not in the ground state space.

$\Rightarrow$ Non-Abelian anyons?

• Non-Abelian anyons can provide a decoherence-free platform to realize quantum computation (**Topological Quantum Computation**)

• Quantum Monte Carlo, however, cannot provide us details from the excitation statistics and no further proofs for non-Abelian anyons.

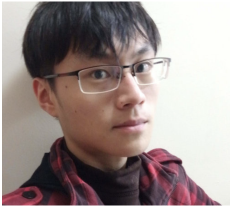
• Exact diagonalization is not useful. The rank of the largest matrix ever diagonalized numerically is 2.7×10^9, corresponding to less than a Hilbert space of a 4×4 lattice.

This is a hard problem: No known analytical methods, and numerics is either impractical or very limited!

Final Comments

- The $U(1)$ Toric Code also presents other rich properties, as
  - ∼ Hilbert space fragmentation
  - ∼ UV/IR mixing
  - ∼ Translation spontaneous symmetry breaking
- **Main conclusion:** Hard problems are hard.
- Currently we are investigating continuum TQFTs consistent with the system that are able to explain the unusual topological degeneracy
- Possibility of experimentally realizing non-Abelian anyons through its connection to the WXY model. See arXiv: 2302.03707 for more!

Collaborators:



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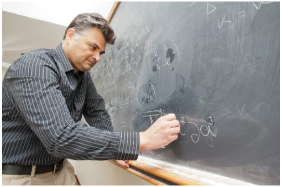

Claudio Chamon