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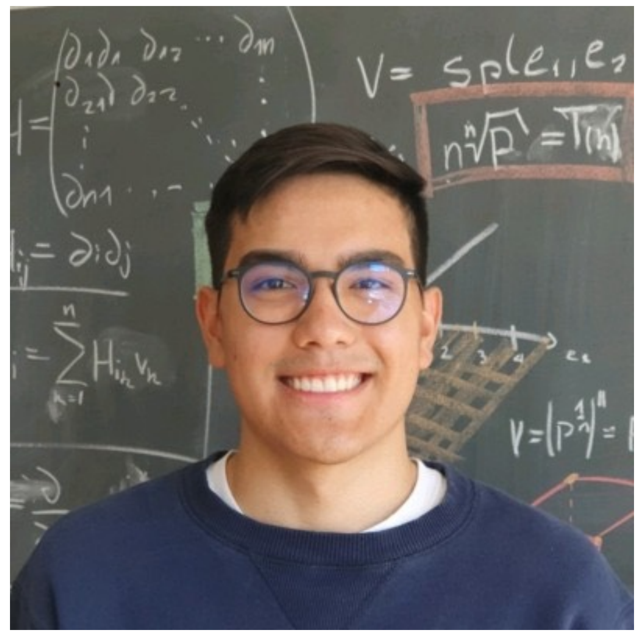
Probing physics in arbitrary graphs and curved spaces

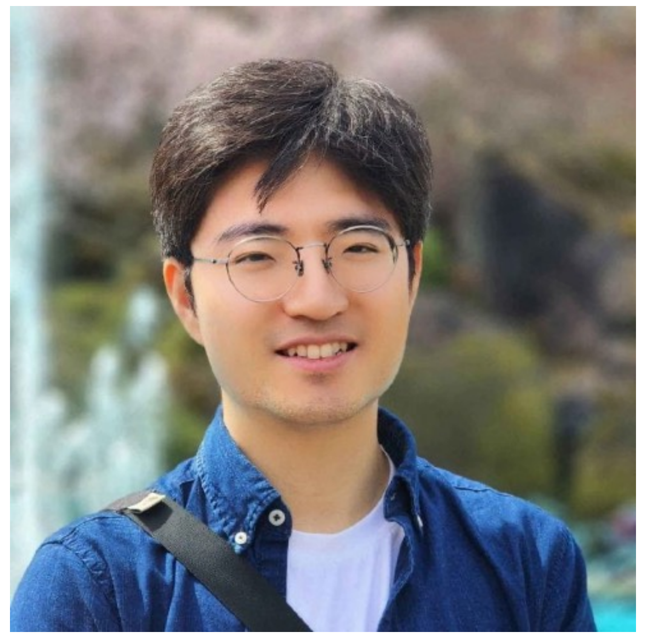
Gui Delfino

arXiv: 2605.28942

Mehmet Dede



André Mudry



Junseok Oh

Andrew Higginbotham

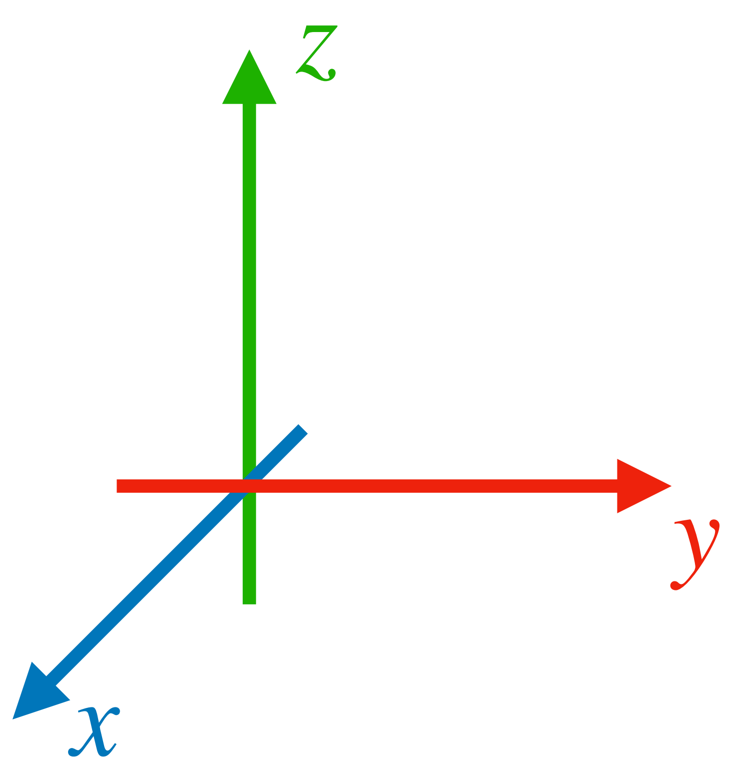
Christopher Mudry

Claudio Chamon

Quantum Matter vs Dimensionality

Gapped quantum phases of matter can come in different flavors depending on space dimension

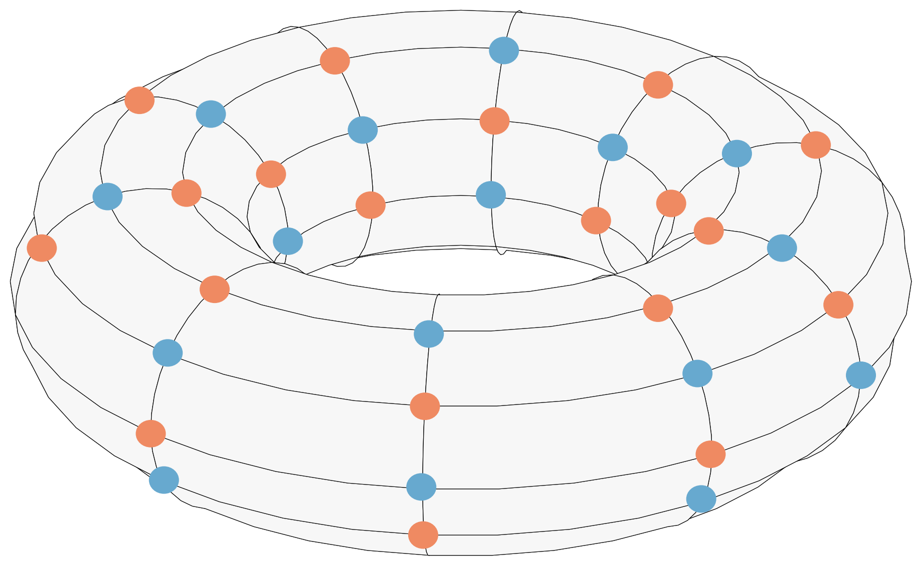
- (1+1)d: Spontaneous symmetry breaking & Symmetry Protected Topological Order (magnets, polyacetylene, topological superconductors, etc)
- (2+1)d: Topological order (quantum spin liquids, FQHE, etc)
- (3+1)d: Fractons
- (4+1)d: Finite temperature stable quantum memories (4D toric code)
- (5+1)d: ??????



- As prisoners to a (3+1)d spacetime jail, we're generally impeded to access such physics

Probing Rich Physics

- Additionally, rich physics may come from curvature / twists in spacetime manifold
  - Non-trivial topologies (topological phases)
  - Curvature (AdS/CFT)



- Locality can be traced back as the source “problem” for such dimensionality and geometry/topology limitations
- Proposal: build such universes by using “non-local” degrees of freedom → Superconducting Wires

Superconducting wires

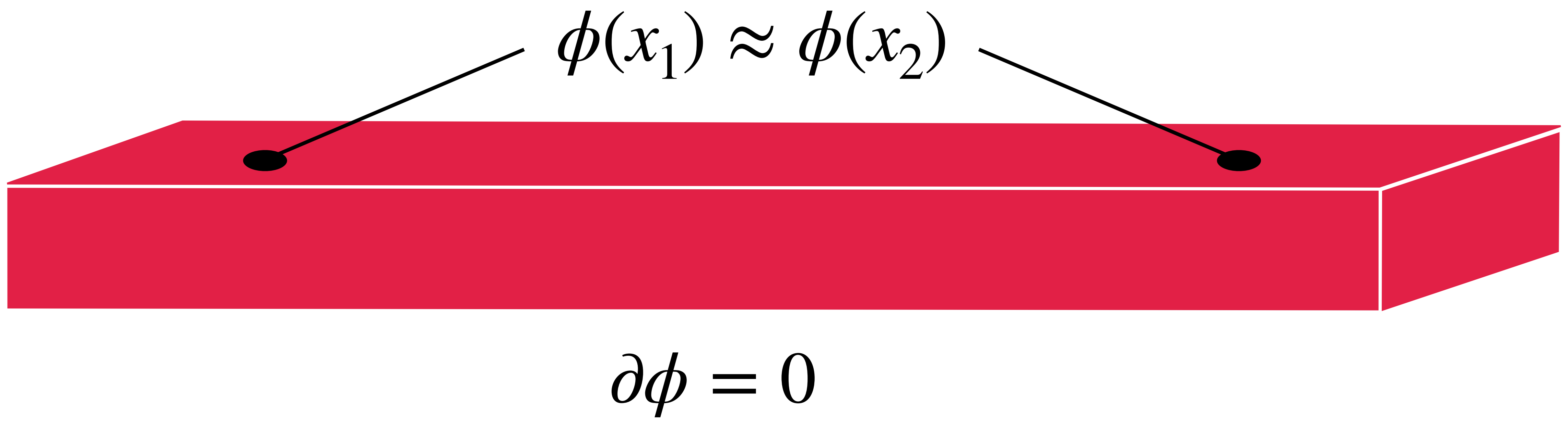
- Superconducting wires are characterized by phase ϕ, which behave as a single, rigid degree of freedom,

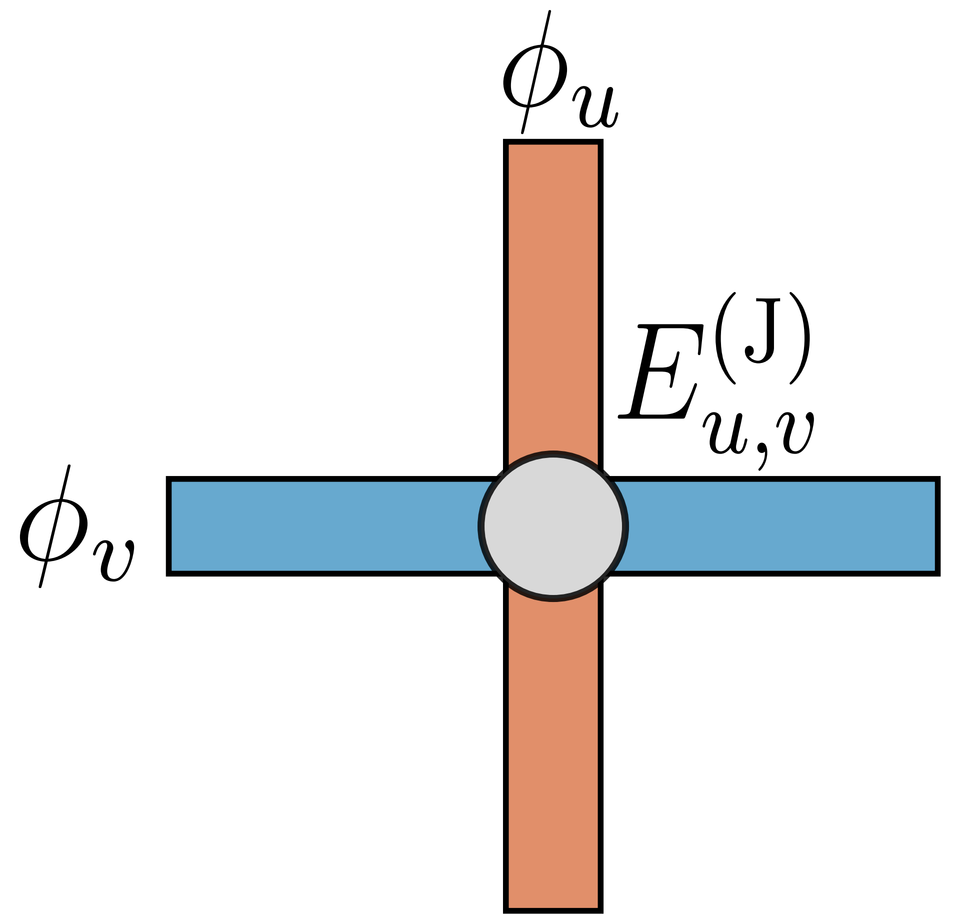
$$\mathscr{F}[\phi] = \frac{1}{2}\int d^3x\, \rho_s\, (\nabla\phi)^2$$

for large stiffness $\rho_s \Rightarrow$ energetic penalty to variations in $\phi(x)$

- Josephson junctions induce energetic interactions of the form

$$H = -\,E^{(\mathrm{J})}_{u,v}\cos(\phi_u - \phi_v)$$





Two Steps

1. Triangulate/put a mash on the manifold M [Cairns (1935) and Whitehead (1939)]

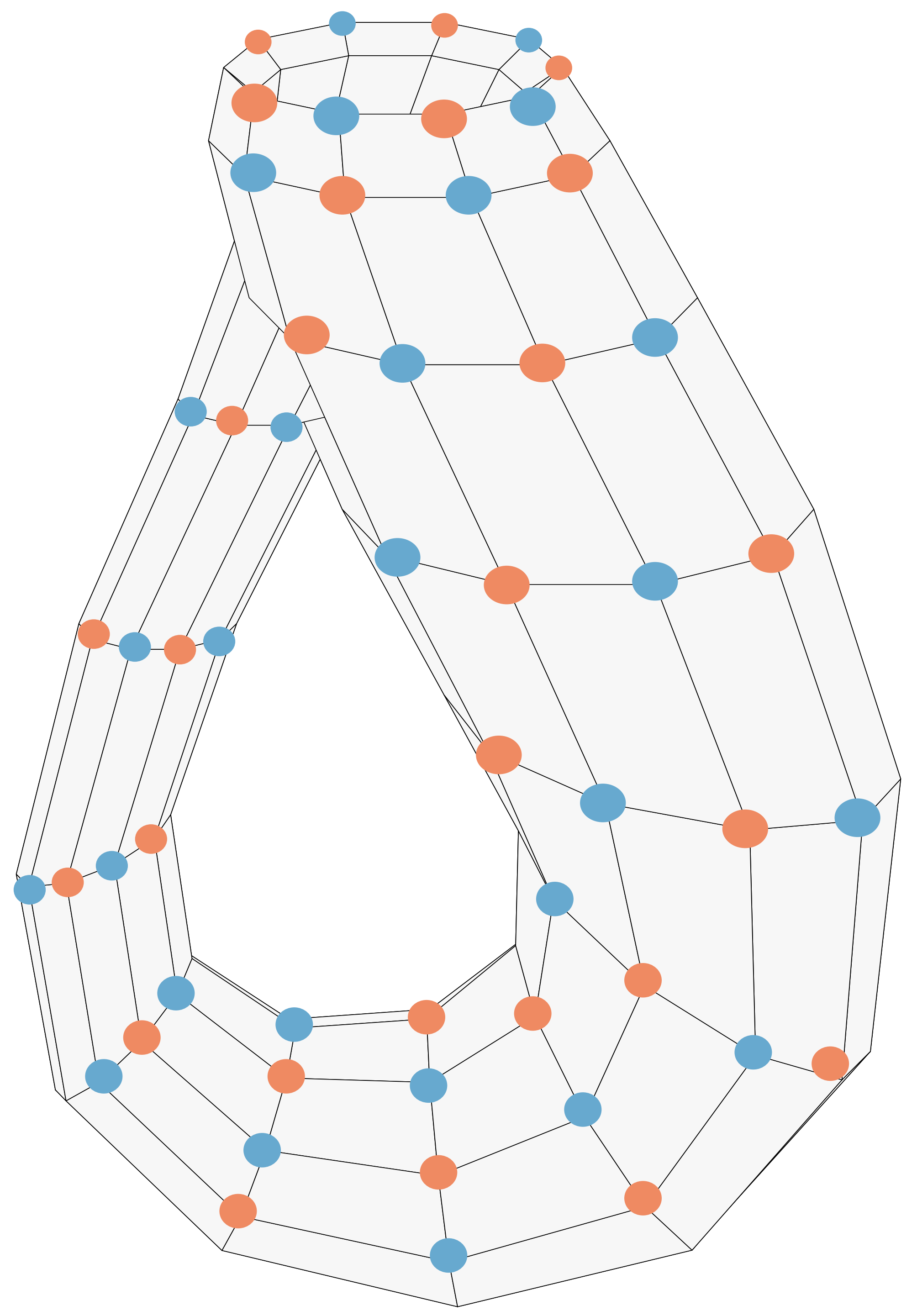
2. Arrange this into a graph structure G = (V,E) on M with

   - Set of nodes (vertices) V

   - Set of links (or edges) E

⇒ The connectivity of the graph can be encoded in a $|V| \times |V|$ adjacency matrix

$$a_{u,v} = 1 \text{ if } \{u, v\} \in E \text{ and } 0 \text{ otherwise.}$$

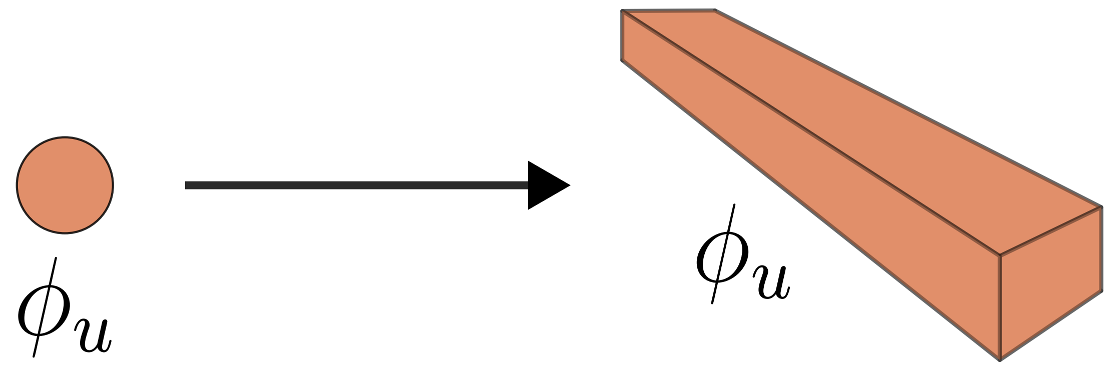


mash for Klein bottle

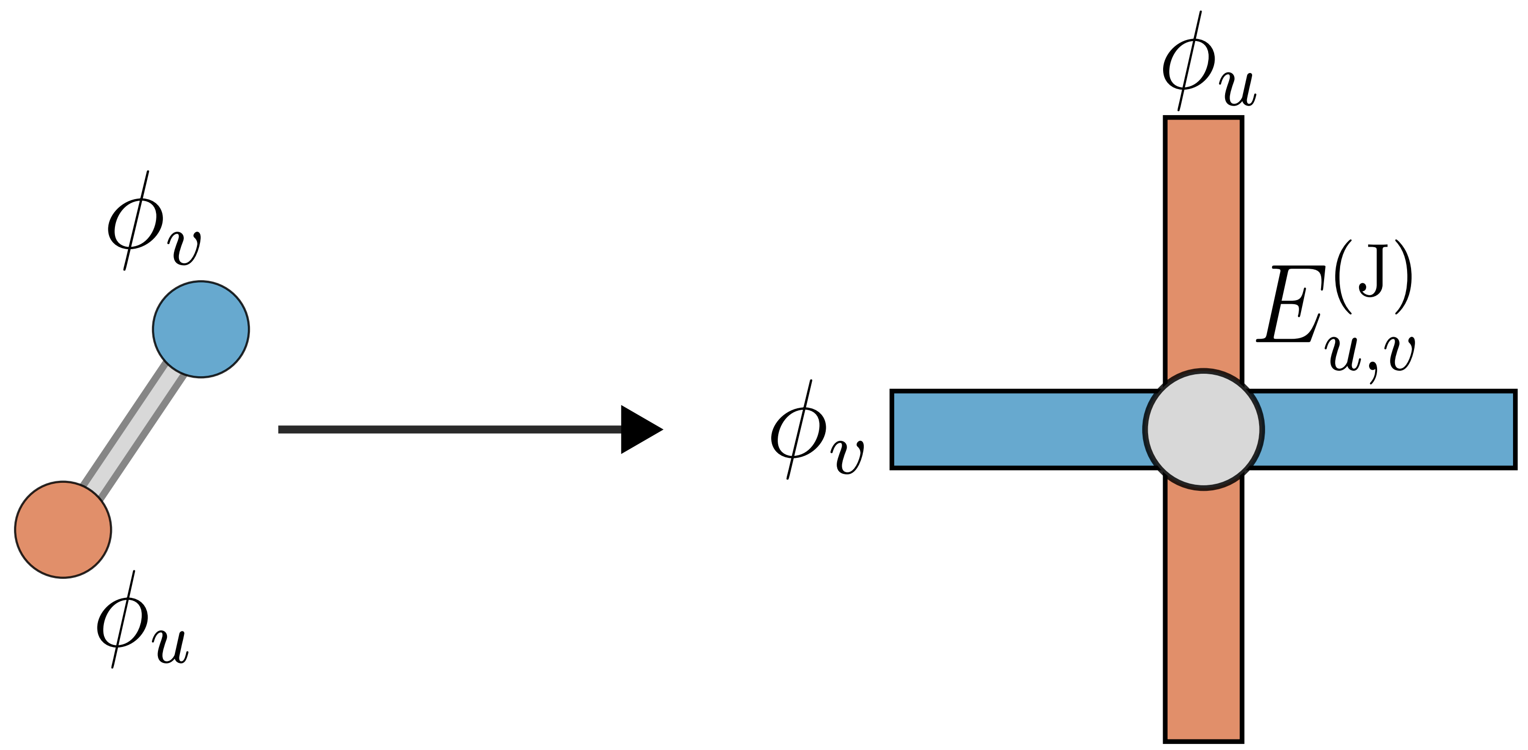
General Framework

Our framework holds for arbitrary simple graphs G

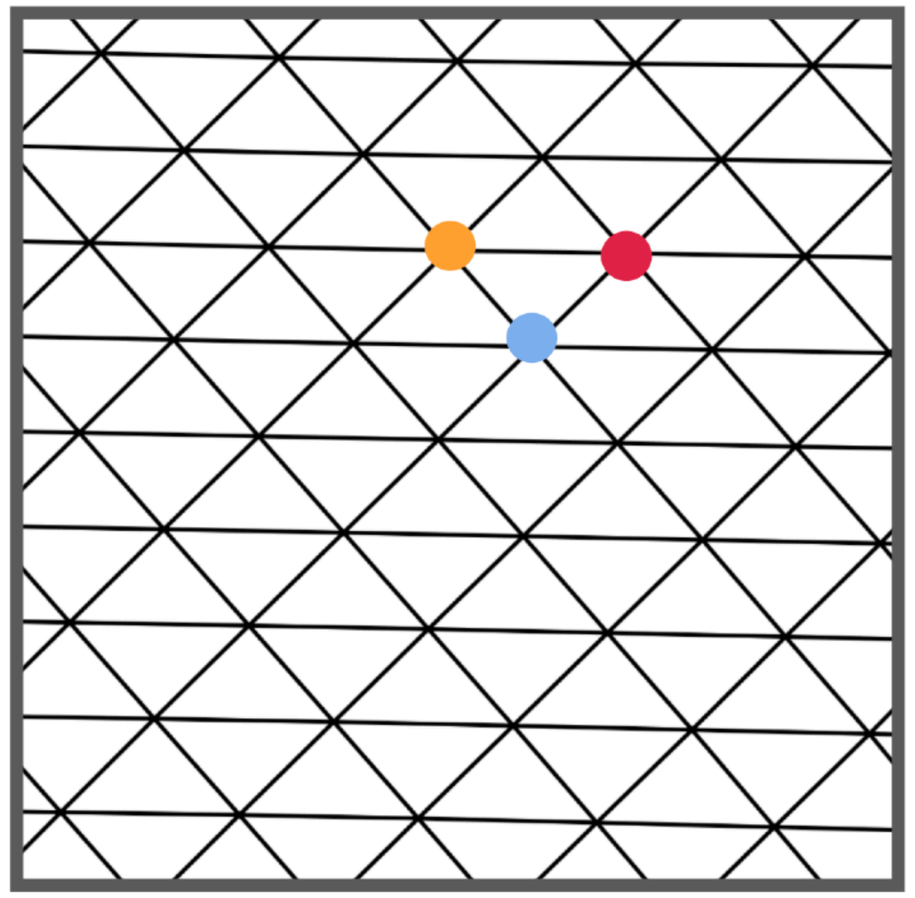
1. To each node $u \in V$, we assign it to a superconducting wire with phase ϕ_u



2. For each edge $\{u, v\} \in E$, we assign it to a Josephson junction with coupling $E_{u,v}^{(J)}$



Bipartite Graphs

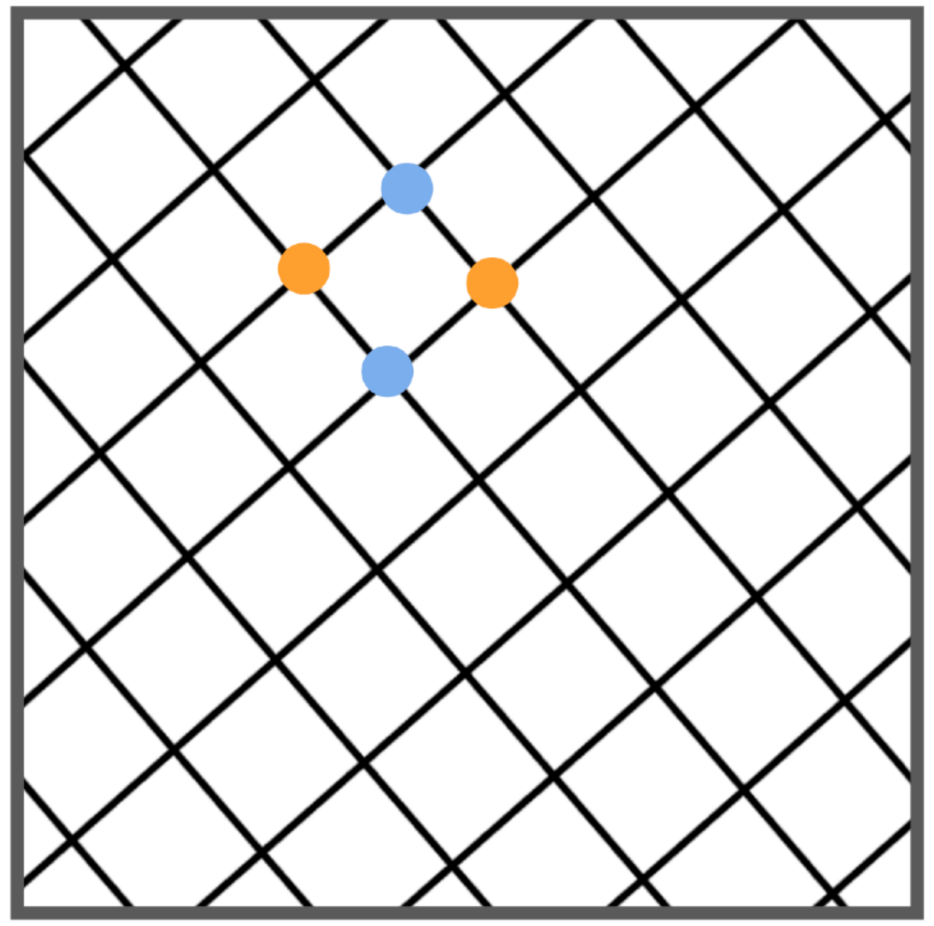


- Although our framework works for arbitrary simple graphs, it simplifies greatly for bipartite ones

- A graph $G = (V, E)$ is said bipartite if $V = V_{\mathrm{orange}} \cup V_{\mathrm{blue}}$

- The adjacency matrix simplifies

$$A = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix}, \qquad \text{with } B \text{ the biadjacency matrix}$$

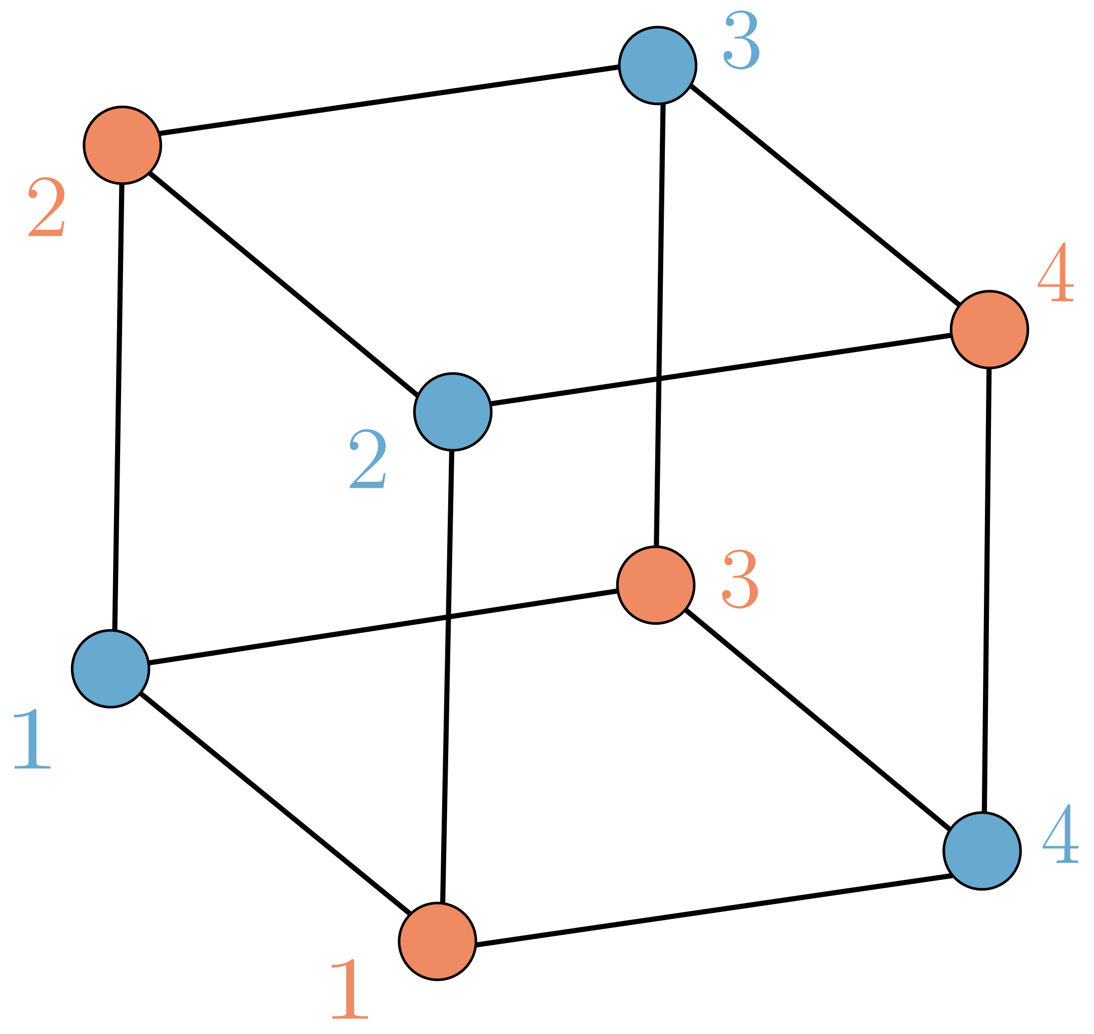


- Then, the SC wire array can be realized through crossing horizontal and vertical wires

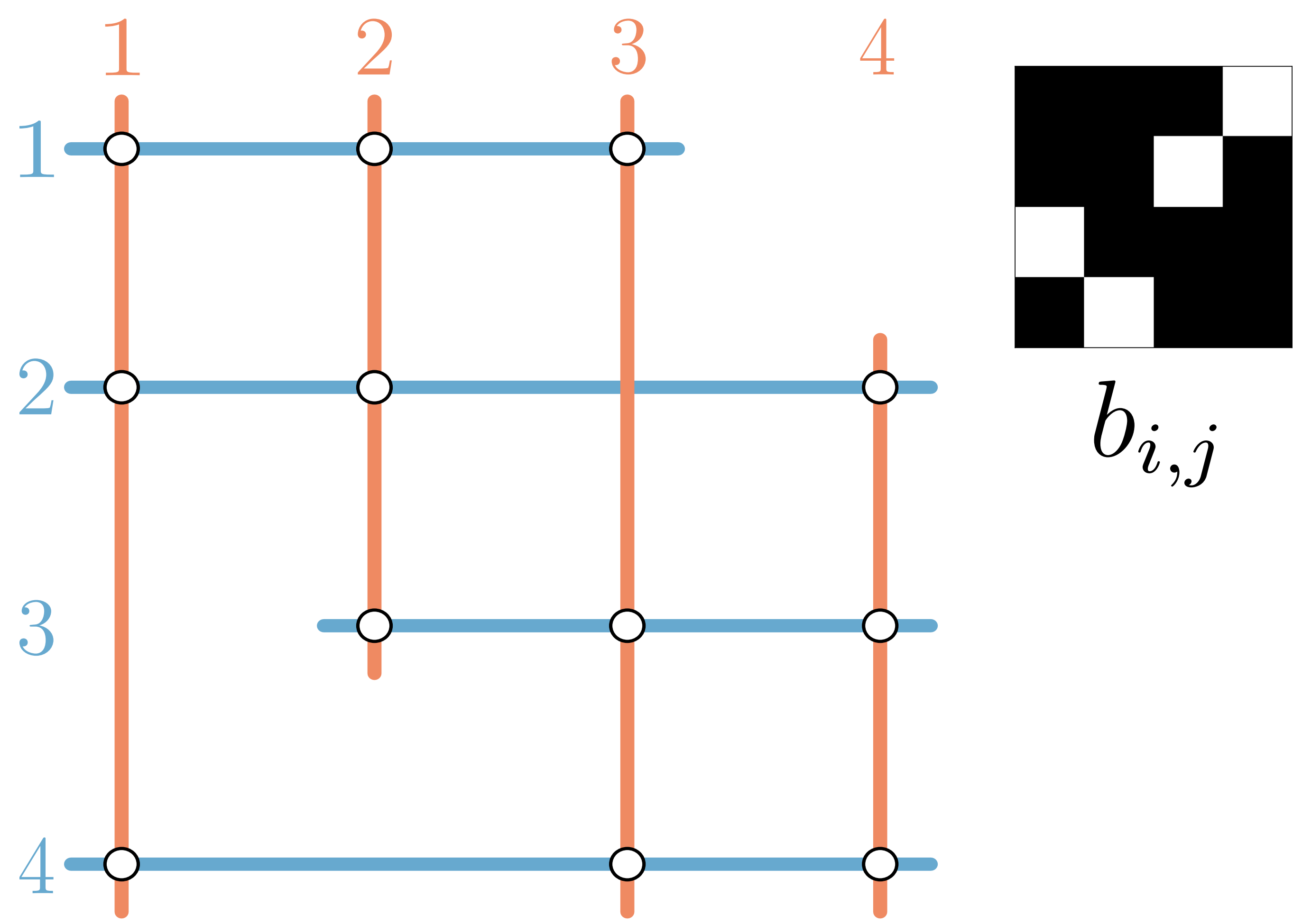
Example: Cube

Under the mapping:

Vertices	⇔	Wires
Edges	⇔	Josephson Junctions
Coordination number	⇔	# of Intersecting wires

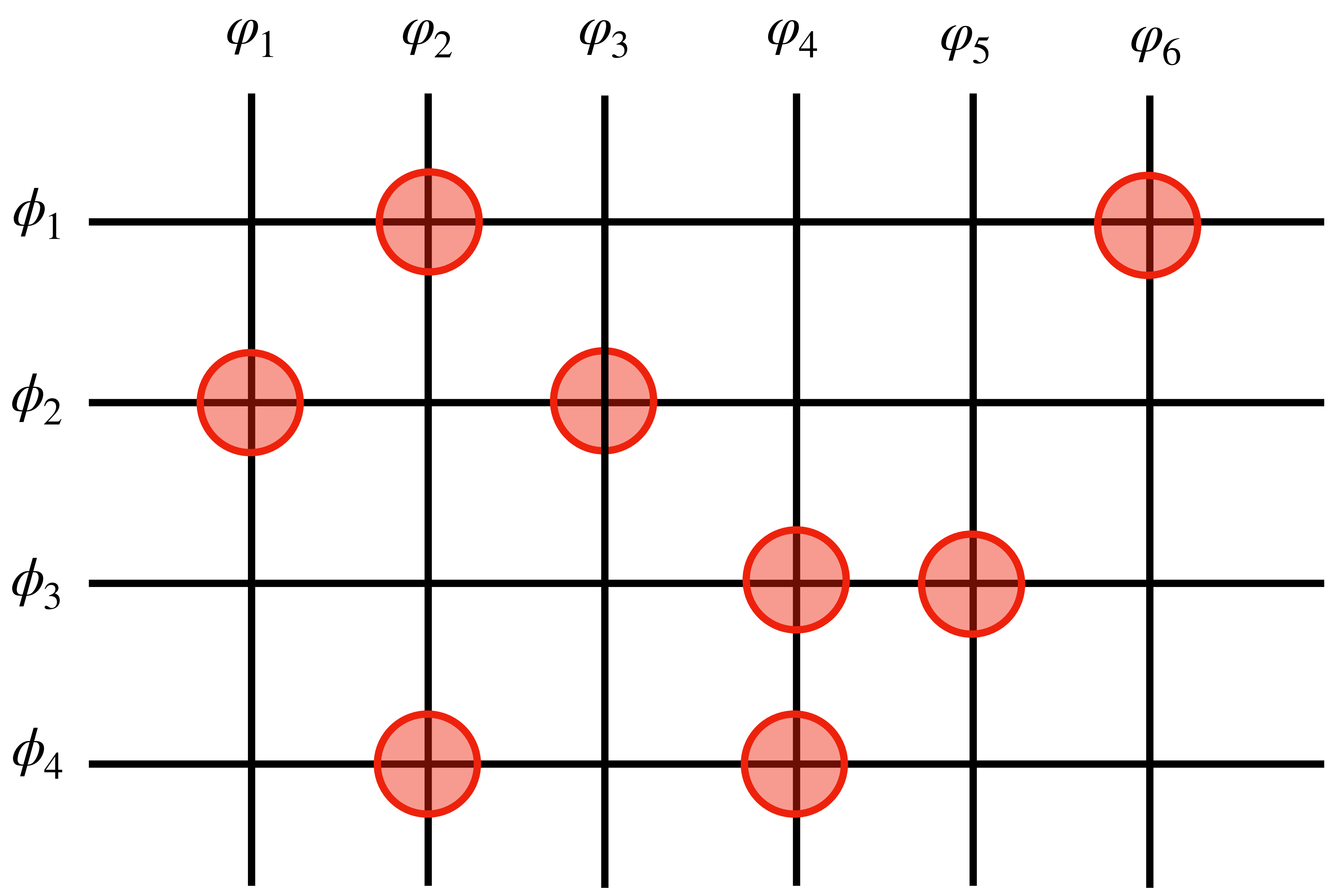


⇔



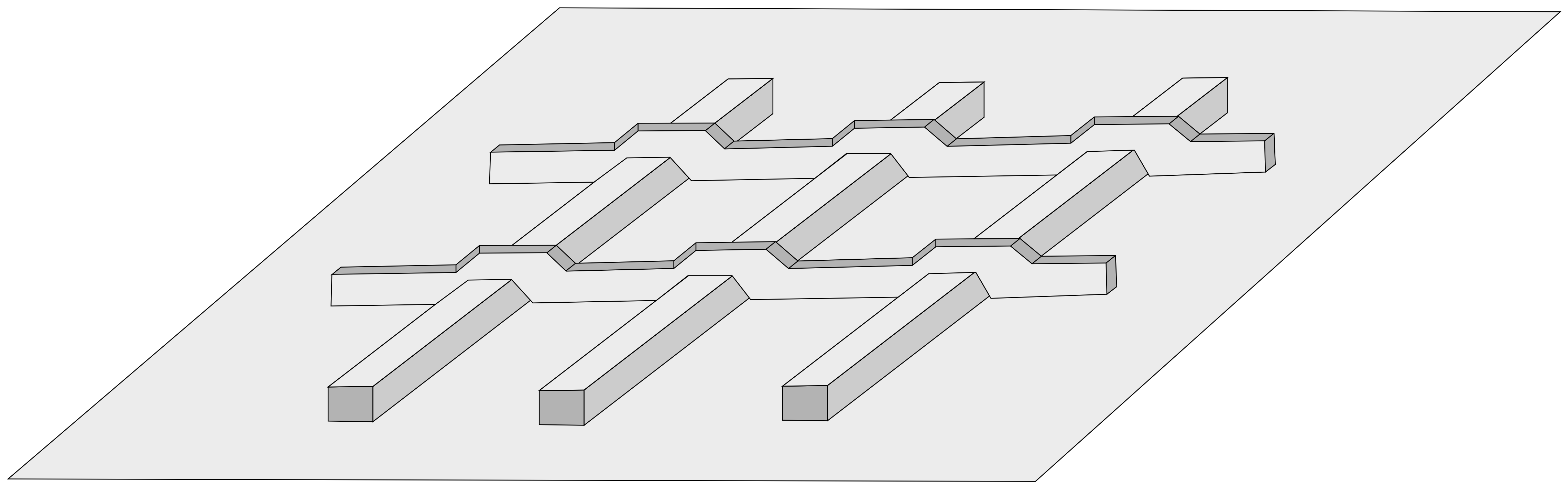
SC wire array as "protoboard"

- The graph $G = (V, E)$ and its biadjacency matrix are the main ingredients in the recipe
- Implementation corresponds to connecting wires together, some version of a "protoboard"



Connectivity matrix $B = (b_{i,j})$

$$H = -J \sum_{i}^{N_1} \sum_{j}^{N_2} b_{i,j} \cos(\phi_i - \varphi_j)$$



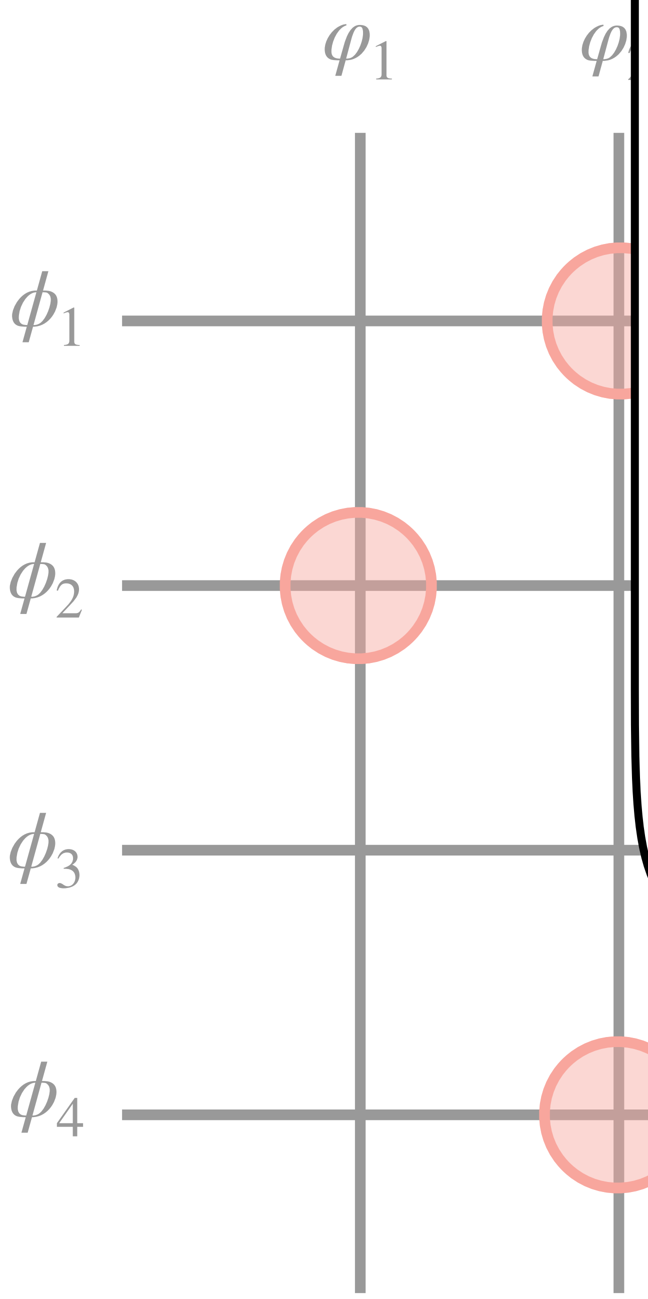
[Sohn et al (1993); Shea & Pinkham (1997)]

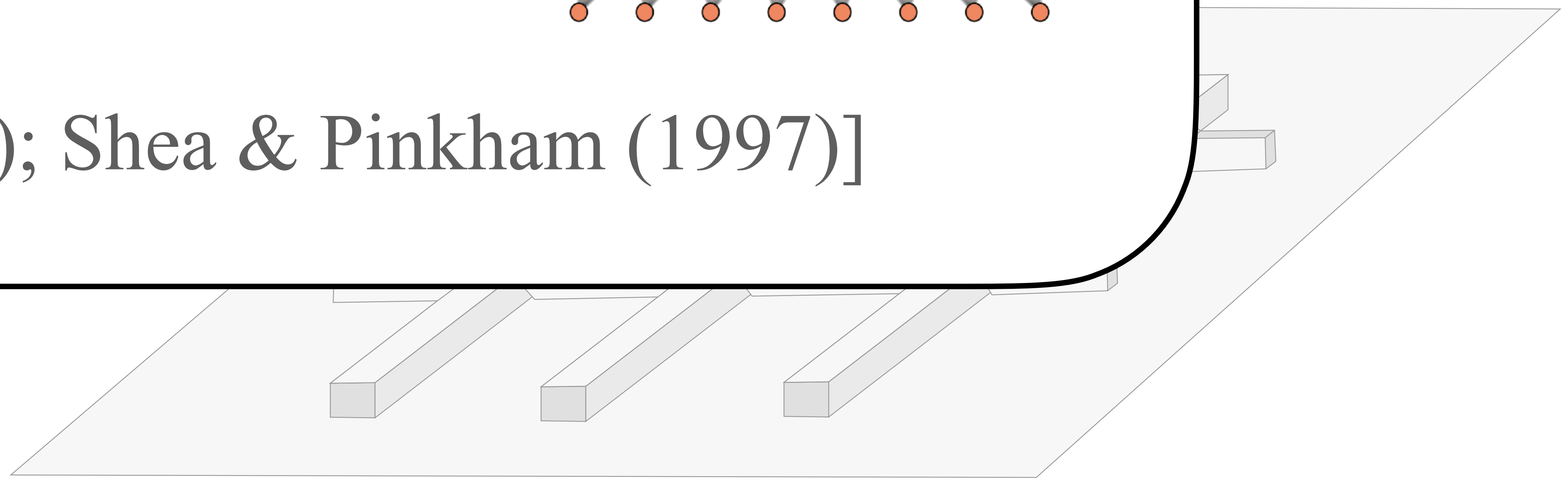
SC wire array as “protoboard”

- The graph ... in the recipe
- Implemen... a “protoboard”

... y matrix $B = (b_{i,j})$

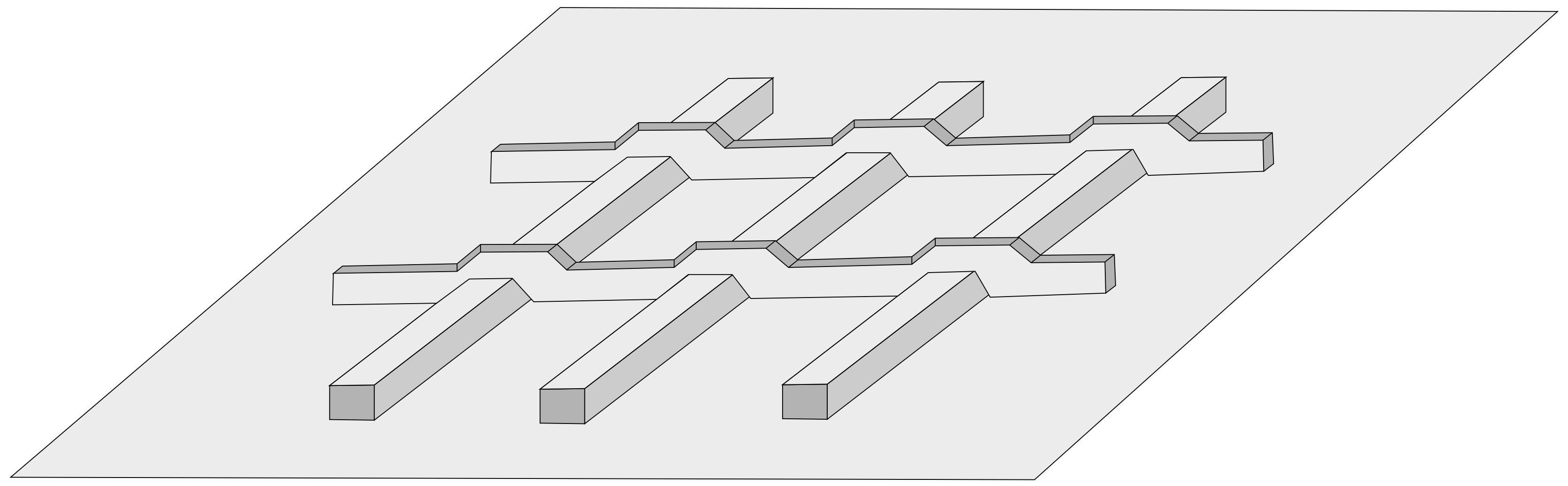
... $\varphi_j)$

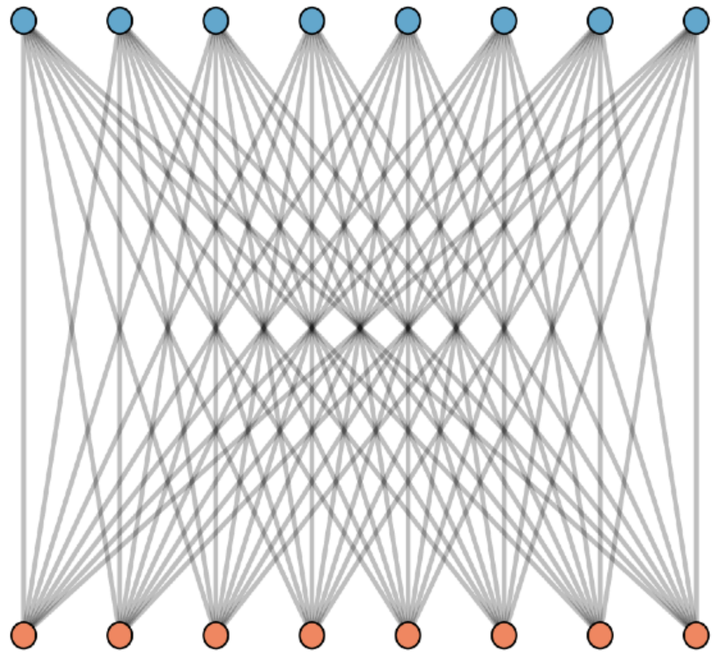




[Sohn et al (1993); Shea & Pinkham (1997)]

For $b_{ij} = 1\,\forall$ pairs $(i, j) \Rightarrow$ Bipartite complete graph $K_{N,N}$

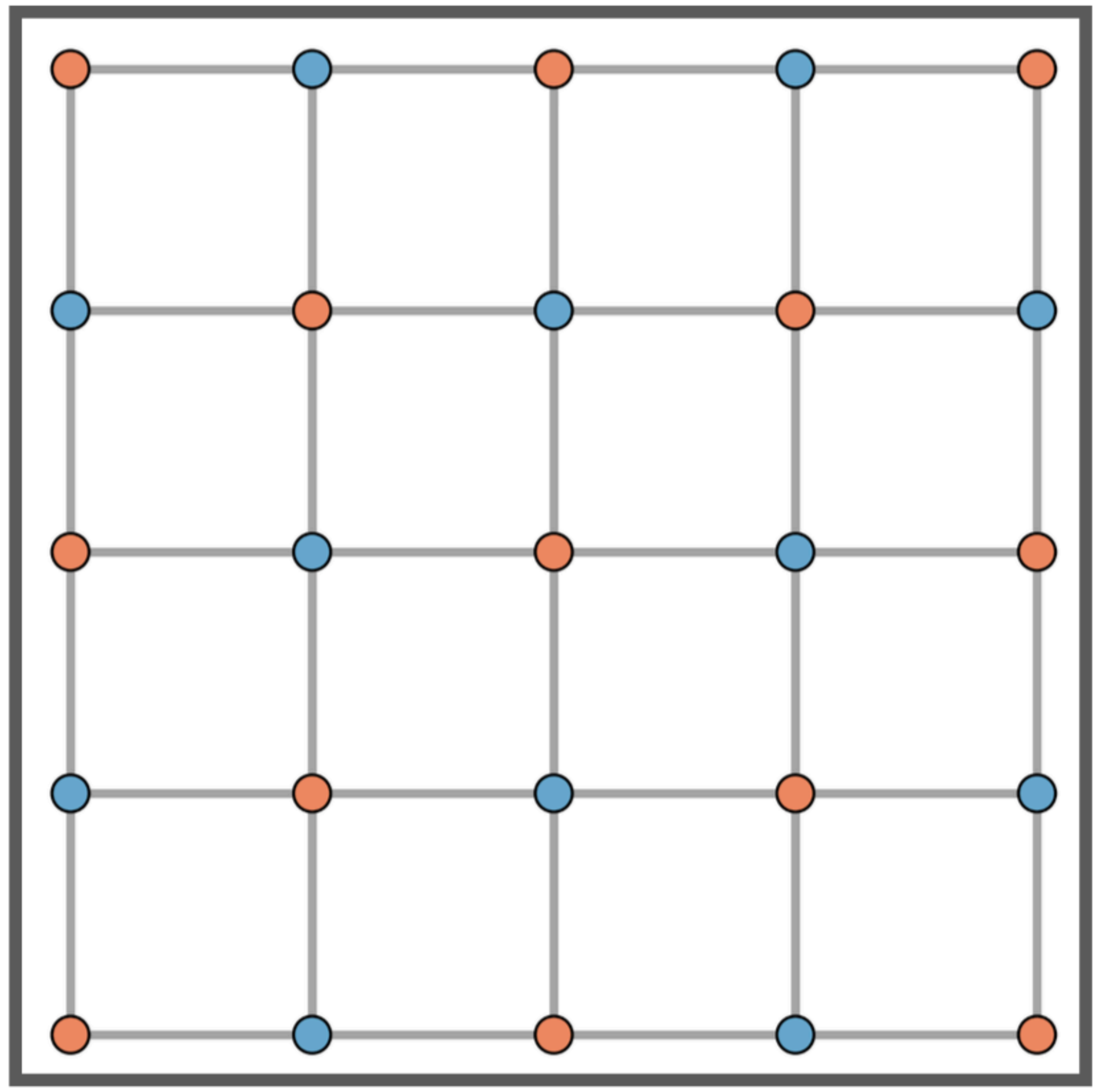




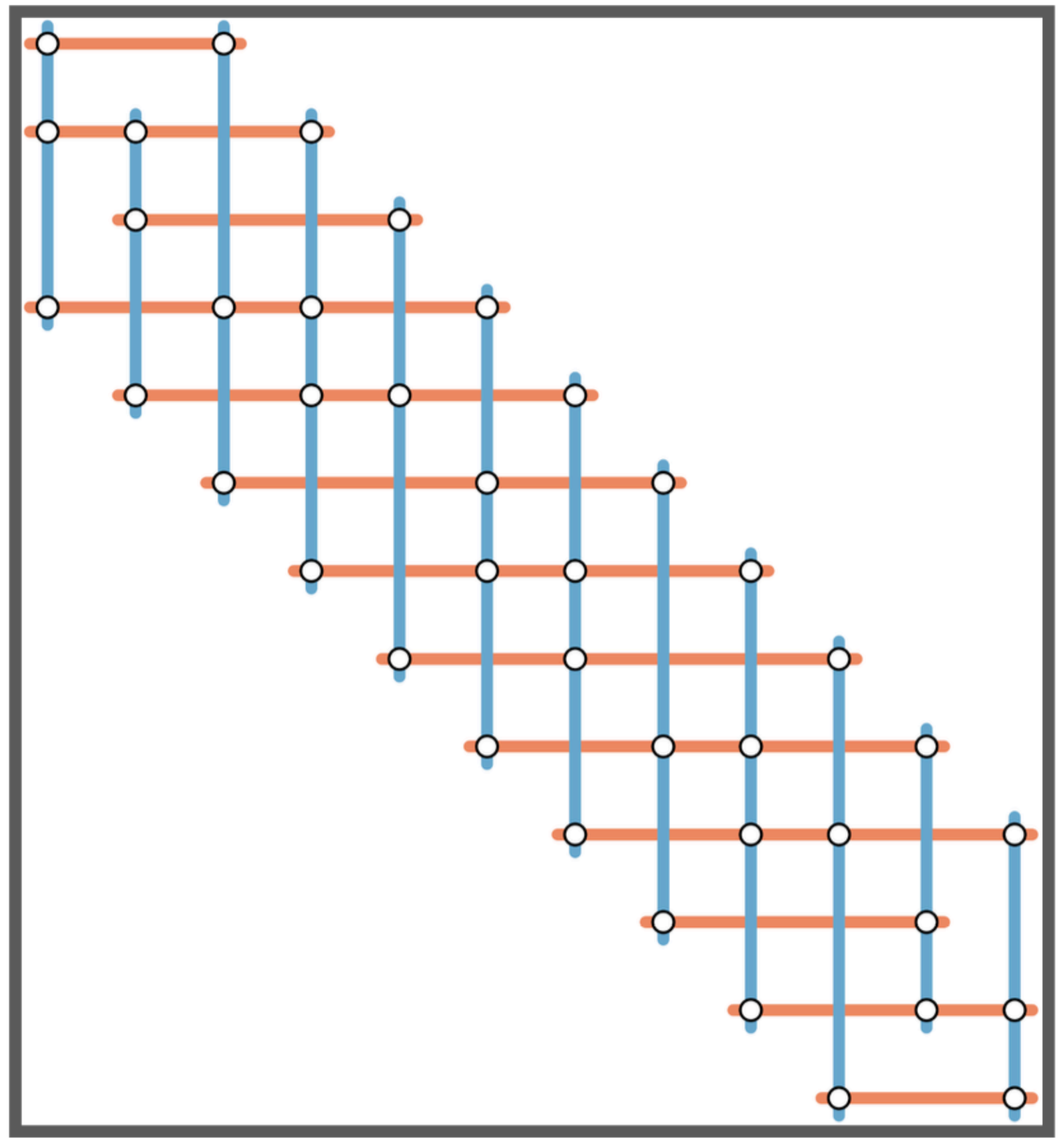
[Sohn et al (1993); Shea & Pinkham (1997)]

Field Theory @ Square Lattice

$$S_G[\phi] = -J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j)$$



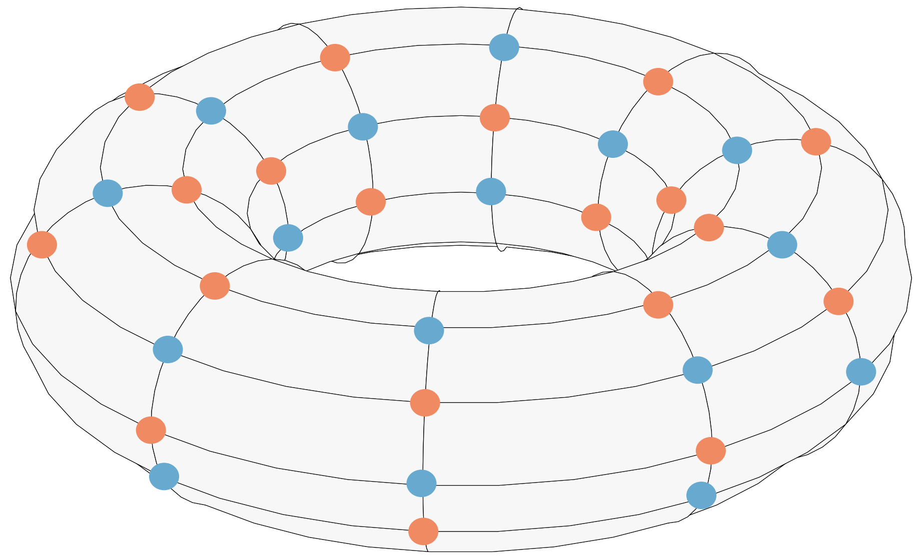
$$S_{\text{wire}}[\phi] = -J \sum_{i \in V_{\text{orange}}} \sum_{j \in V_{\text{blue}}} b_{i,j} \cos(\phi_i - \phi_j)$$

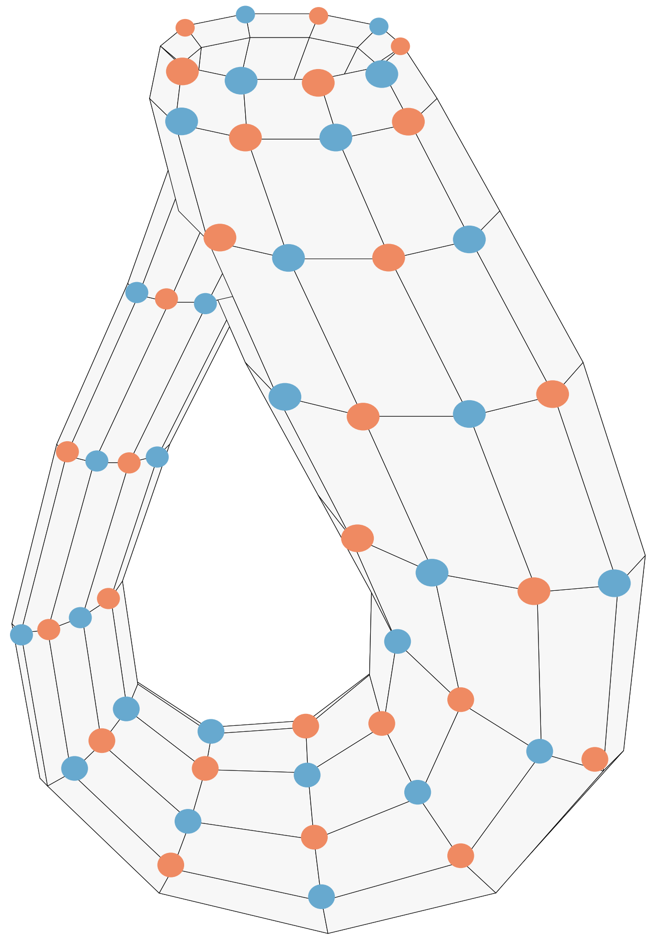


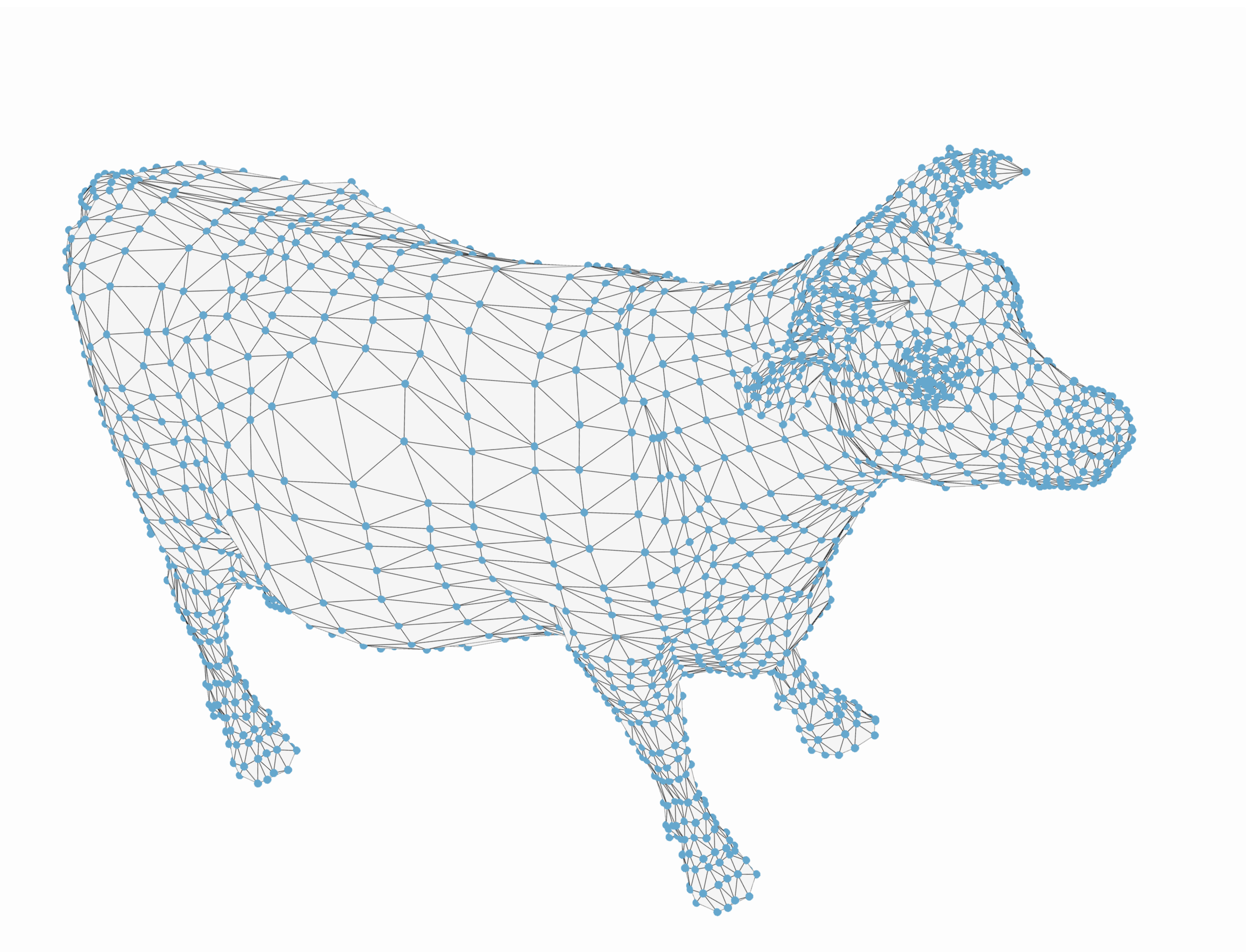
$$S_G[\phi] = S_{\text{wire}}[\phi]$$

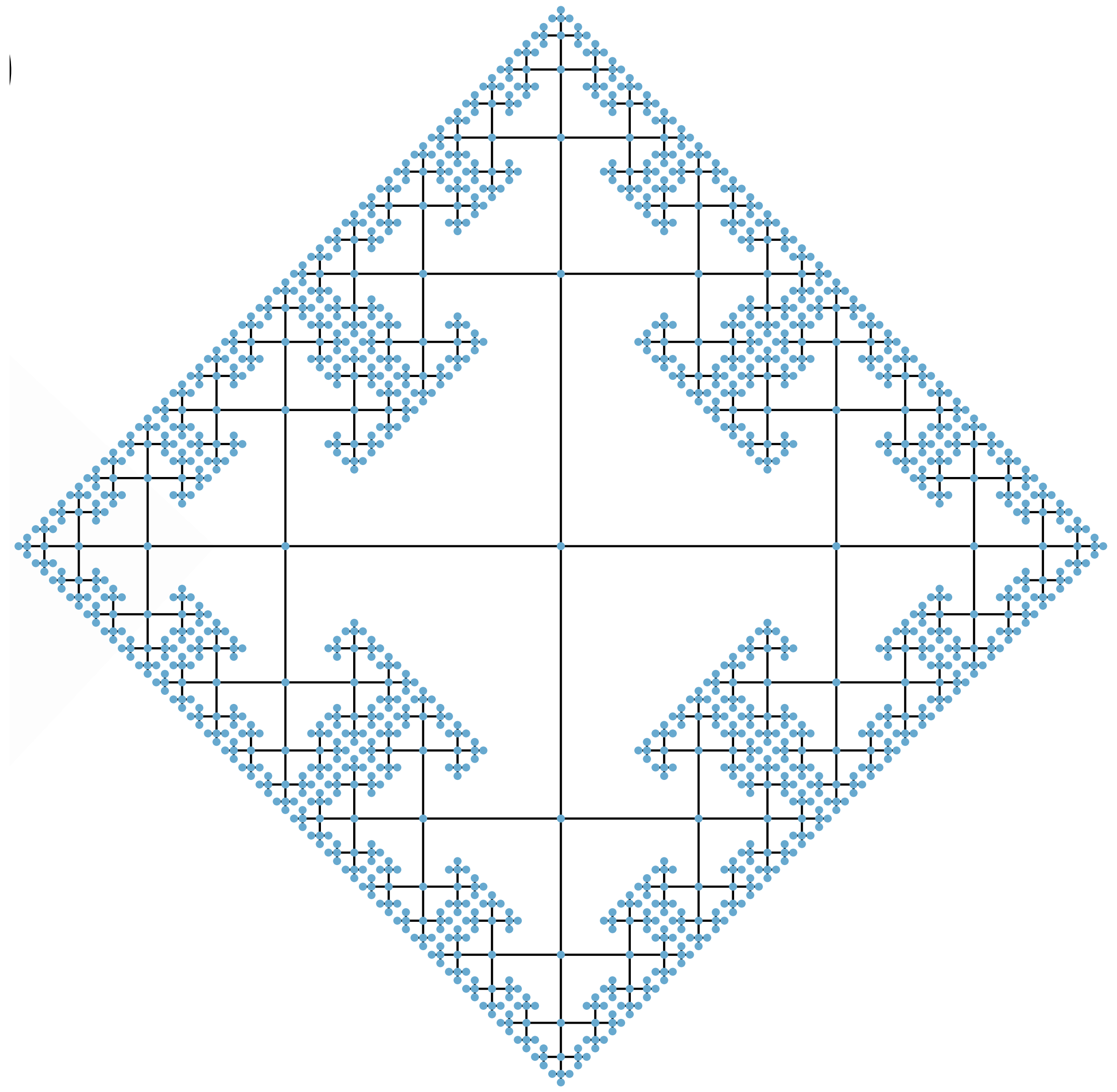
Arbitrary Simple Graphs

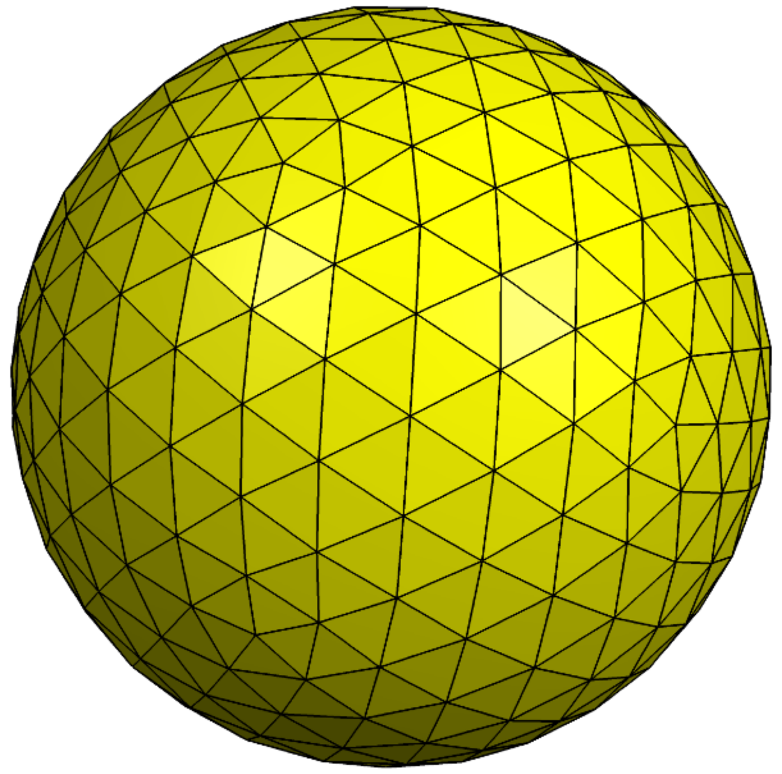
- Higher Dimensions
- Curved Spaces
- Cayley graphs for abstract groups
- Etc











Quantum Fluctuations

In general, arrays of Josephson junctions have quantum fluctuations $\left[\hat{\phi}_u, \hat{n}_v\right] = \mathrm{i}\,\delta_{u,v}$ induced by charge operators $\hat{n}_v$. Then $\widehat{H} = \widehat{H}_{\mathrm{J}} + \widehat{H}_{\mathrm{C}}$

$$\widehat{H}_{\mathrm{J}} := -\frac{1}{2}\sum_{u,v} E^{(\mathrm{J})}_{u,v} \cos\left(\hat{\phi}_u - \hat{\phi}_v - A^{(\mathrm{bg})}_{u,v}\right),$$

$$\widehat{H}_{\mathrm{C}} := \frac{1}{2}\sum_{u,v} E^{(\mathrm{C})}_{u,v} \left(\hat{n}_u - n^{(\mathrm{bg})}_u\right)\left(\hat{n}_v - n^{(\mathrm{bg})}_v\right),$$

$E^{(\mathrm{J})}_{u,v}$ is the Josephson energy and $E^{(\mathrm{C})}_{u,v}$ is the charging energy between wires u and v, where $E^{(\mathrm{C})}_{u,v}$ is usually set by geometry (wire size).

$\Rightarrow$ Here we focus on the classical limit $E^{(\mathrm{C})}_{u,v} \to 0$

Validity of framework

- The framework is limited by wire sizes

$$E_\pi \approx \frac{1}{2}\rho_s AL\left(\frac{\pi}{L}\right)^2 \text{ and } N = \frac{L}{a} = \frac{L}{\alpha w}$$

- Using realistic experimental parameters $\rho_s^{\mathrm{Al}} \approx 10^{-12}\,\mathrm{J/m}$, $A \approx (200\,\mathrm{nm})^2$, with $E_\pi \approx 1\mathrm{K}$ (10x bigger than experiment temperature $100\,\mathrm{mK}$)

- $L \approx 14\,\mathrm{mm}$, which allows us to cross $N \approx 3.5 \times 10^4$ wires spaced by $400\,\mathrm{nm}$ each

Field Theory @ Anti-de-Sitter spaces

- $\mathrm{AdS}_{\mathrm{d+1}}$ spaces are defined on hyperbolic spaces $\quad \mathrm{d}s^2 = \dfrac{\mathrm{d}z^2 + \mathrm{d}\mathbf{x}^2}{z^2}, \quad z > 0, \quad \mathbf{x} \in \mathbb{R}^d$

  with a conformal boundary at $z \to 0$

- Consider the massive scalar field coupled to the above background metric g on $\mathbb{H}^{\mathrm{d+1}}$.

$$S[\phi] = \frac{1}{2} \int_{\mathbb{H}^{d+1}} \mathrm{d}^{d+1}X \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + m^2 \phi^2 \right),$$

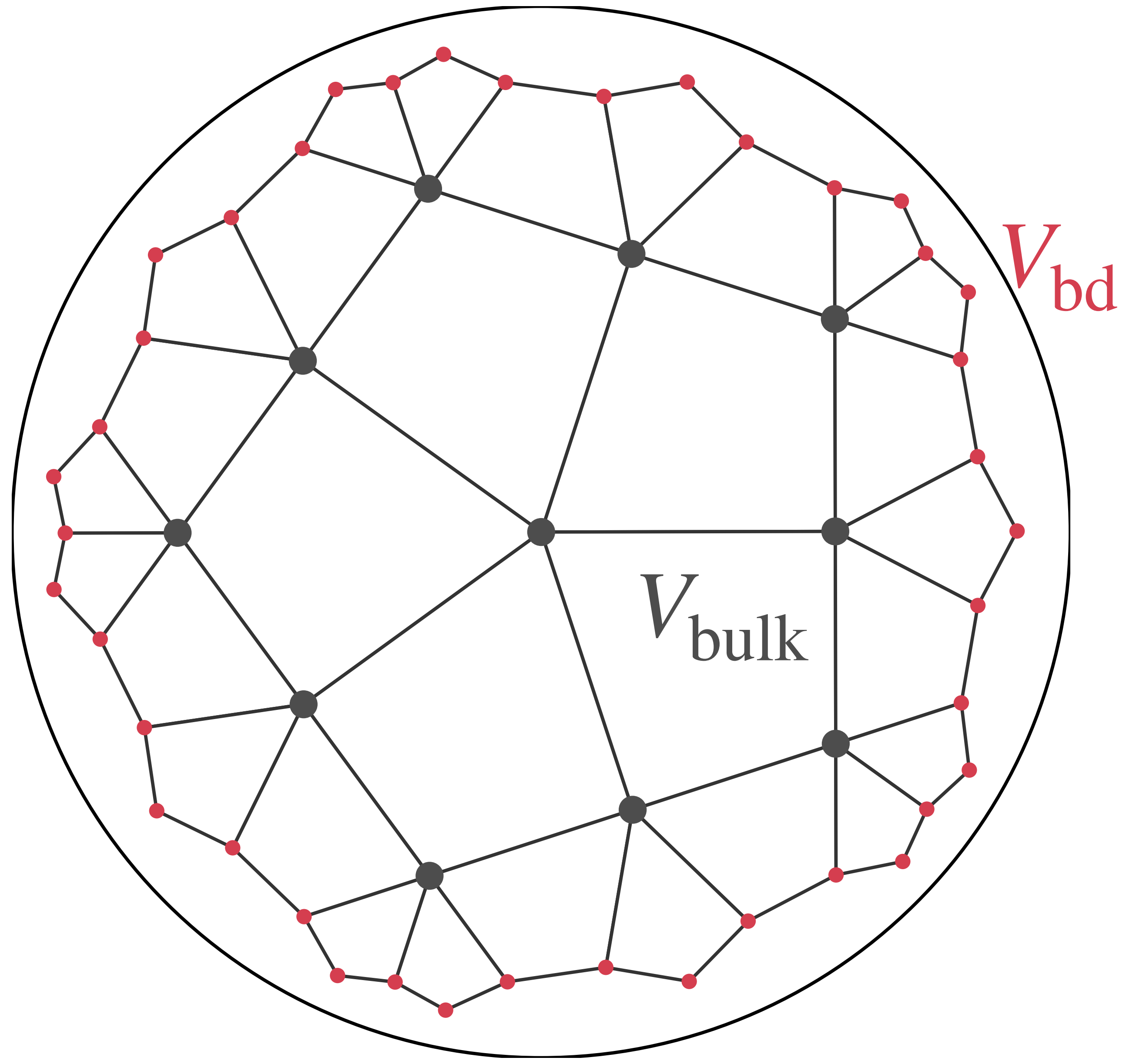
- Boundary theory displays power-law decay $\langle \mathcal{O}(\mathbf{x_1}) \mathcal{O}(\mathbf{x_2}) \rangle = \dfrac{c}{|\mathbf{x_1} - \mathbf{x_2}|^{2\Delta}}$, with

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{2} + m^2}$$

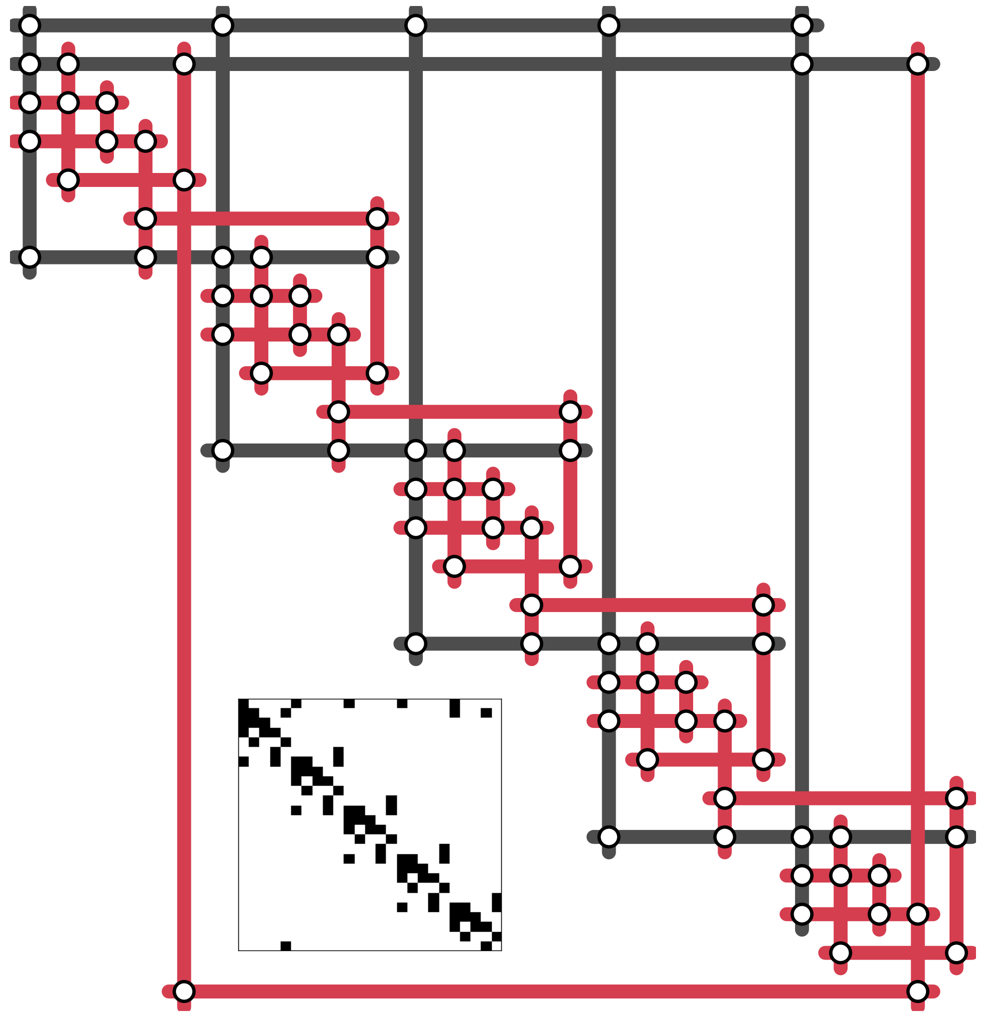
[Witten (1998)]

Setup

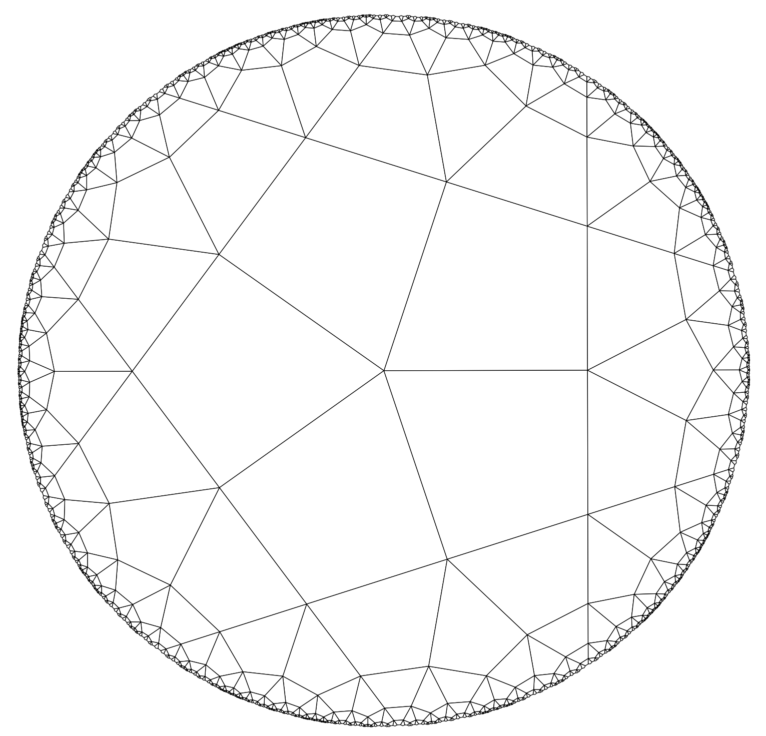
- Let us consider the $\{4,5\}$ discretization of $\mathbb{H}_2$ place in Poincare-disk coordinates $G = (V, E)$, with $V = V_{\text{bulk}} \cup V_{\text{bd}}$



- To account for mass terms, we add a “big” superconductor ϕ_∞ that biases all other phases



$$H_G = \frac{1}{2} \sum_{u,v \in V} a_{u,v}\, J\, [1 - \cos(\phi_u - \phi_v)] + \sum_{u \in V} m^2\, [1 - \cos(\phi_u - \phi_\infty)] + \sum_{u \in V_{\text{bd}}} M^2\, [1 - \cos(\phi_u - \phi_\infty)]\,,$$



Linearization

- Assuming that $m^2 \gg J$ then $\phi_u - \phi_v \ll 1$ and we linearize

$$H_G = \frac{1}{4} \sum_{(u,v)\in E} J_{u,v} \left(\phi_u - \phi_v\right)^2 + \frac{1}{4} \sum_{u\in V} \left(m^2 + M^2 \delta_{u\in V_{\mathrm{bd}}}\right) \phi_u^2 + \dots$$

$A = (a_{u,v})$
adjacency matrix

- We define the quadratic action $S[\phi, J] = \frac{1}{2} \sum_{u,v} \phi_u L_{u,v} \phi_v$ with Laplacian

$$\delta_{u\in V_{bd}} = \begin{cases} 1, & u \in V_{bd} \\ 0, & u \in V_{bulk} \end{cases}$$

$$L_{u,v} = -J\, a_{u,v} + d_u\, \delta_{u,v} + (m^2 + M^2 \delta_{u\in V_{\mathrm{bd}}}) \delta_{u,v} \ ,$$

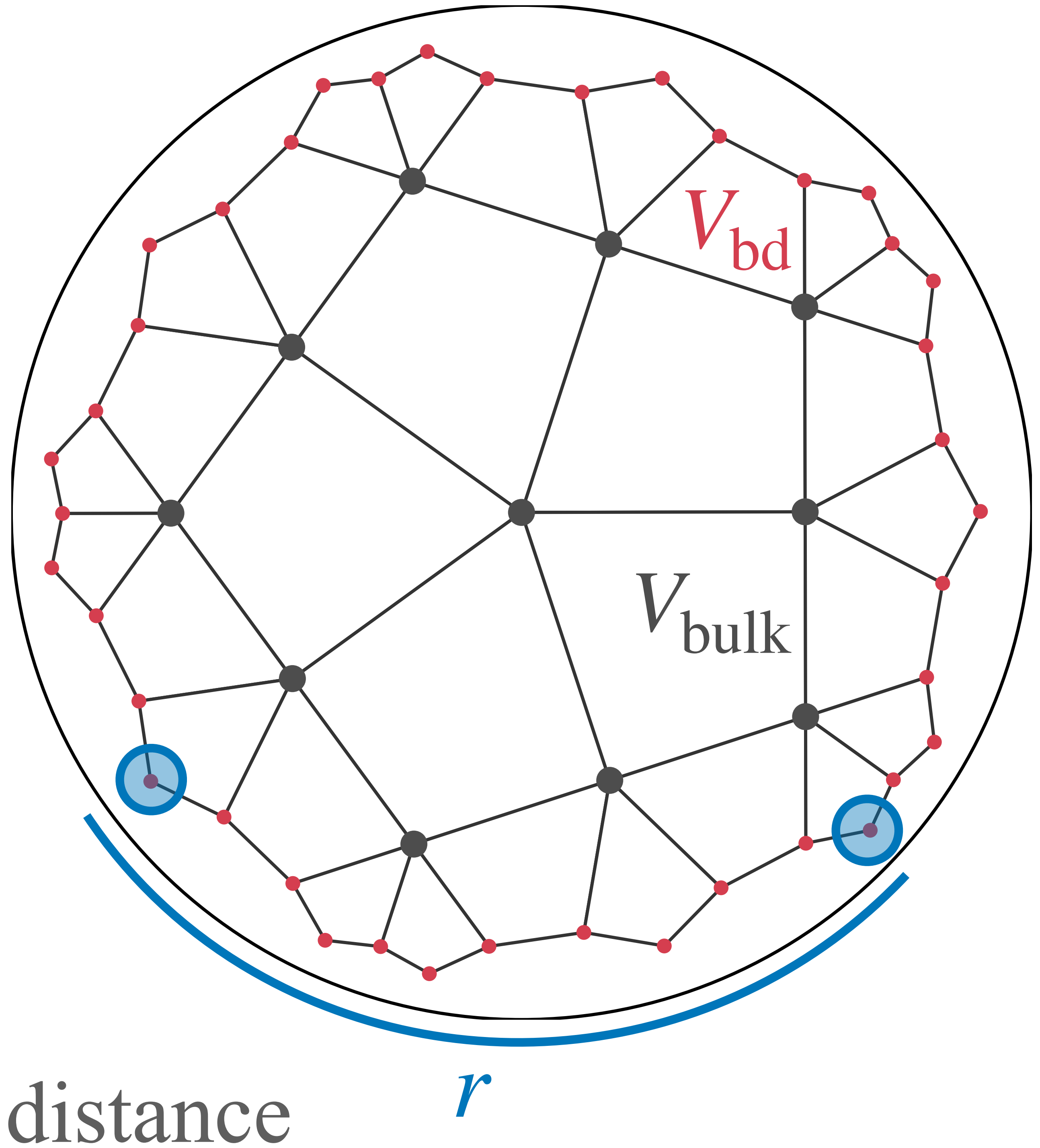
- Additionally, we fix $M^2 \gg m^2$ to enforce Dirichilet boundary conditions

$$d_u = J \sum_v a_{u,v}$$

Boundary-boundary propagators

- Let us consider boundary points and compute edge propagators

$$G^{(0)}_{\text{edge}}(r) \equiv \frac{\sum_{u,v\in V_{\text{bd}}} \left(L_{u,v}\right)^{-1} \delta_{r,d(u,v)}}{\sum_{u,v\in V_{\text{bd}}} \delta_{r,d(u,v)}}, \quad \text{for } u, v \in V_{\text{bd}}$$



Where $d(u, v)$ is the distance between u and v along the boundary

From conformal invariance, we expect

$$G^{(0)}_{\text{edge}}(r) \sim \left[\frac{1}{2\ \sin^2\left(\pi\, r/r_{\max}\right)}\right]^{\Delta} \approx r^{-2\Delta}$$

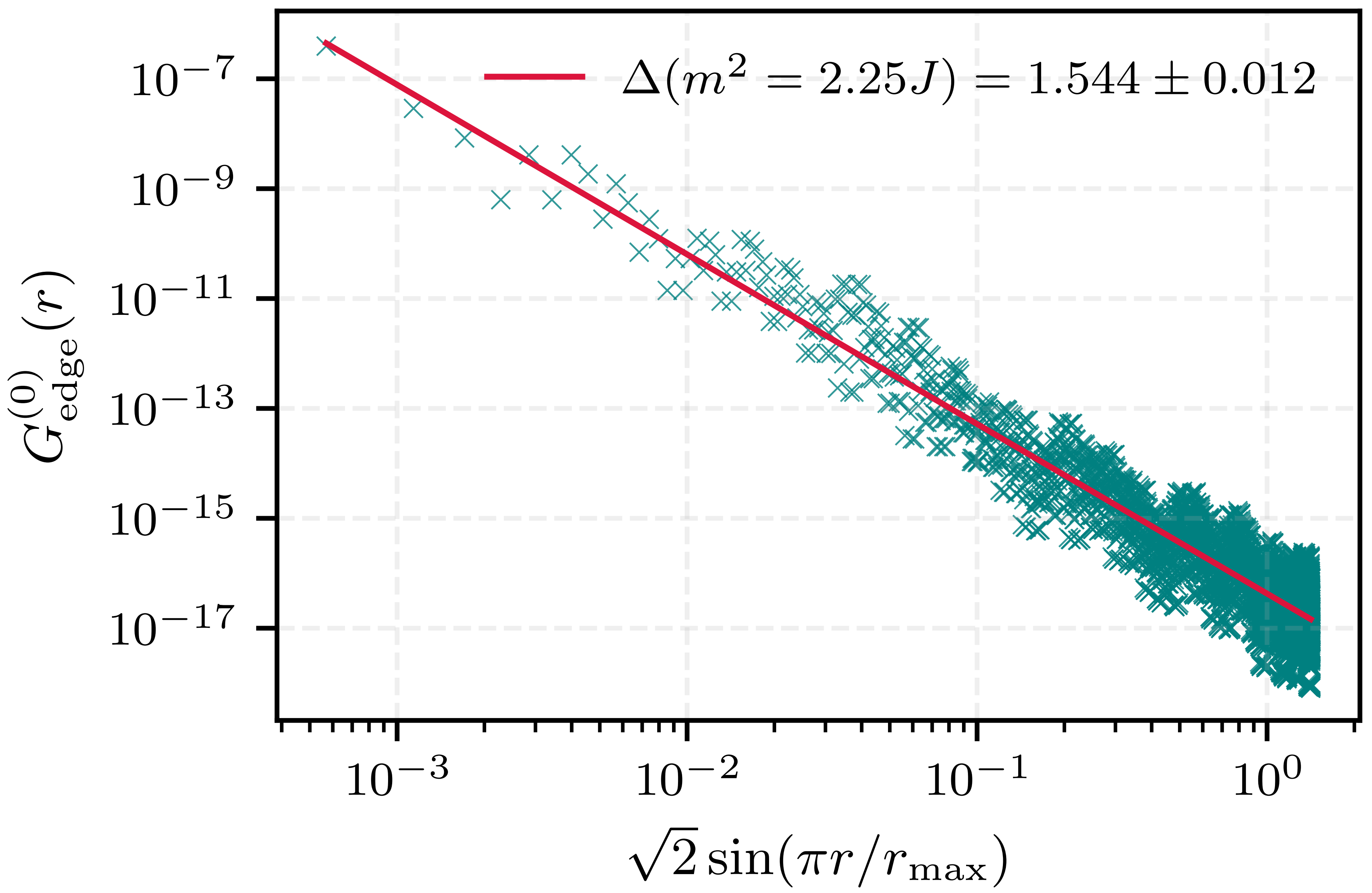
r: boundary distance

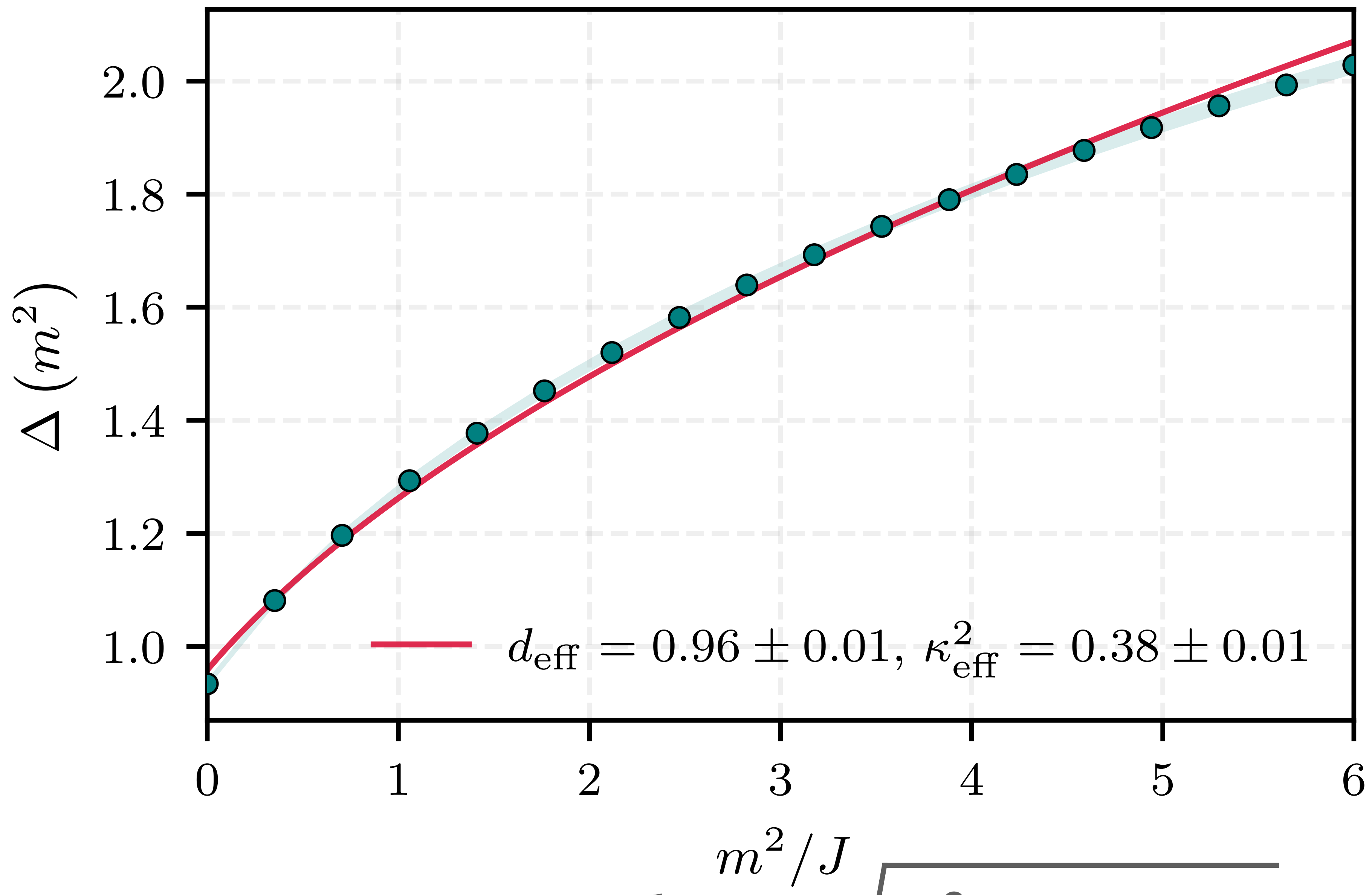
$r_{\max}$: perimeter of boundary

$2\sin^2(\pi r/r_{\max})$: conformal chordal distance

Numerical Results

- We observe a straight line, indicating the power-law behavior with coefficient $2\Delta(m)$





$\Rightarrow$ How stable are the results against imperfections?

$$\Delta(m) = \frac{d_{eff}}{2} + \sqrt{\frac{d_{\mathrm{eff}}^2}{4} + \kappa_{\mathrm{eff}}^2\, m^2}$$

[Assaduzzaman et al (2020)]

Disordered Josephson Junctions

- To account for disorder we treat $J_{u,v}$ as random variable with average J and variance σ^2
    - Gaussian distribution $J_{u,v} \sim \mathcal{N}(J, \sigma^2)$
    - There are no-vanishing probabilities of having $J_{u,v} < 0$: negative bond terms coming from random flux through wire plaquettes.
    - We're interested in the average boundary-boundary correlator

$$\overline{G}_{\text{edge}}(r) := \frac{\sum\limits_{u,v \in V_{\text{bd}}} \delta_{r,d(u,v)} \overline{\langle \phi_u \phi_v \rangle}}{\sum_{u,v \in V_{\text{bd}}} \delta_{r,d(u,v)}}$$

Numerical Estimator

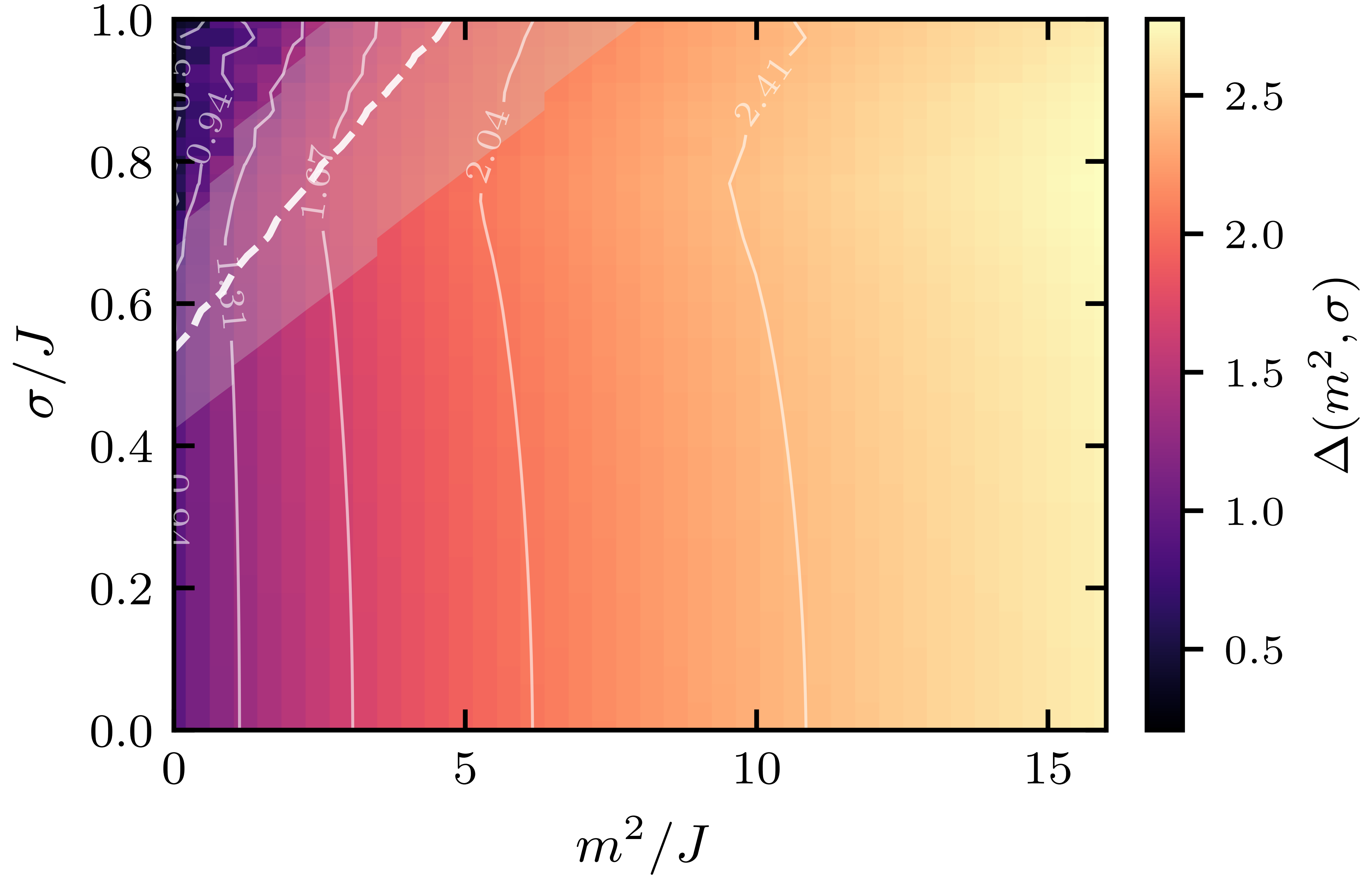
- For valid values of σ, the boundary-boundary correlator obeys a power law with scaling $\Delta(m^2, \sigma)$

$$\overline{G}_{\text{edge}}(r) \sim r^{-2\Delta(m^2,\sigma)}$$

- Breakdown for $\sigma \gg m^2$: Laplacian is not positive definite

- We check that $\Delta(m^2, \sigma = 0) = \Delta(m^2)$

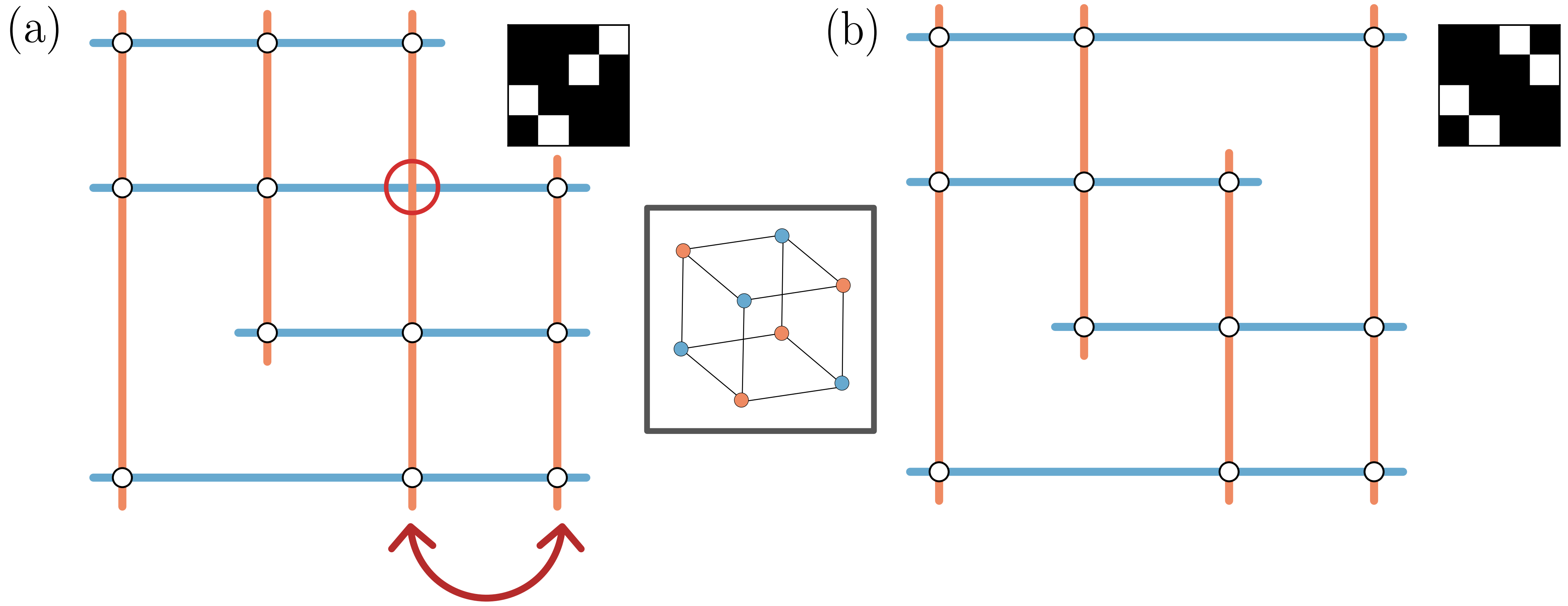
- For small σ deviation from the clean case is mild; which increases continuously for larger σ



Avoiding Overpasses

- Experimentally, it's convenient to avoid unnecessary crossings, which may induce undesired couplings

- This can be done by permutation of wires

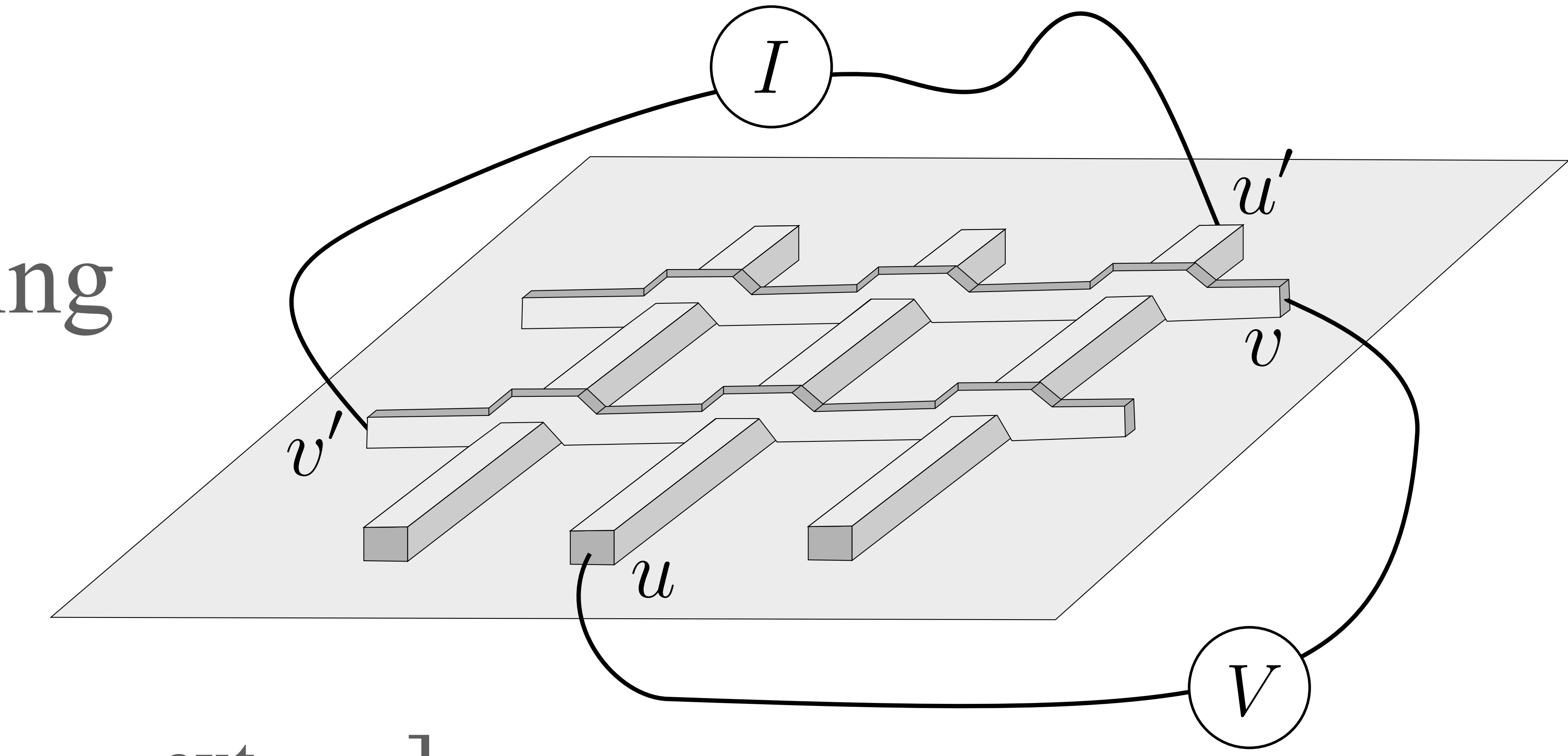


⇒ For planar graphs, a representation with no undesired overpasses is always guaranteed to exist.

In general, deciding whether a graph admits such a representation is NP-hard [Kratochvil (1994)]

Characterization

- Probing correlation functions is equivalent to measuring linear response functions



$$\left\langle \left[\hat{n}_u(\omega) - \hat{n}_v(\omega)\right] \right\rangle_{0\,\beta} = \sum_{u',v'} \chi_{\beta;uv,u'v'}(\omega)\left[I^{\text{ext}}_{u'}(\omega) - I^{\text{ext}}_{v'}(\omega)\right]$$

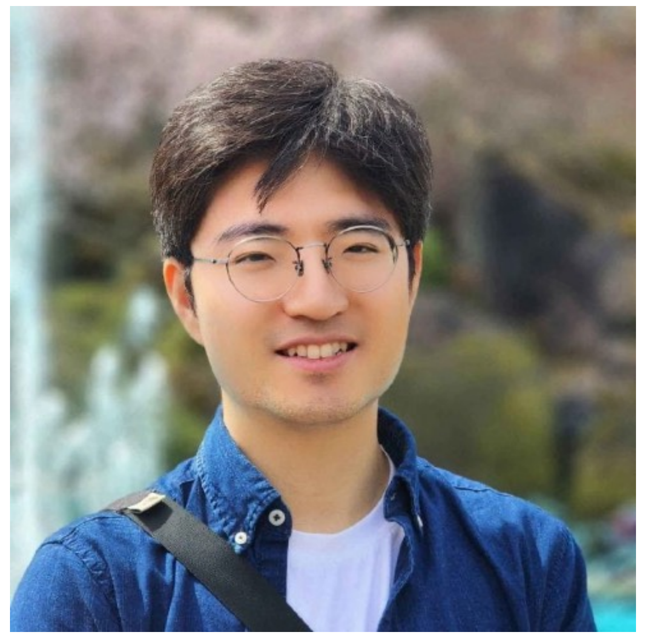
β: Temperature

ω: AC frequency

with the equality

$$\lim_{\beta\to\infty} \lim_{\omega\to 0} \chi_{\beta;uv,u'v'}(\omega) = +\frac{\mathrm{i}}{\hbar} \lim_{\beta\to\infty} \left\langle \left(\hat{\phi}_u - \hat{\phi}_v\right)(0)\left(\hat{\phi}_{u'} - \hat{\phi}_{v'}\right)(0) \right\rangle_{0\,\beta}.$$

Final Considerations



Junseok Oh

Andrew Higginbotham

- Experimental Realization
  - Fabrication method: Manhattan process (two-angle deposition)
  - Al wires (few mm each) with AlOx insulating barriers at $T = 0.1K$
- Future Perspectives:
  - Explore quantum fluctuations: $E_{u,v}^{(\mathrm{C})}$ as "charge connectivity matrix"
  - Implement it in different experimental platforms (LRO)
  - Probe non-local physics (bosonic SYK-like-models) & other geometries;

Thank you!

gui-delfino@github.io