Aspects of Spatially Modulated Symmetries

Guilherme Delfino

BOSTON UNIVERSITY

U.S. DEPARTMENT OF ENERGY
Office of Science

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Introduction

Symmetries are a powerful tool in theoretical physics and can constrain the low energy physics of physical systems, which can be

- Spontaneously broken
- Anomalous
- Gauged
- …

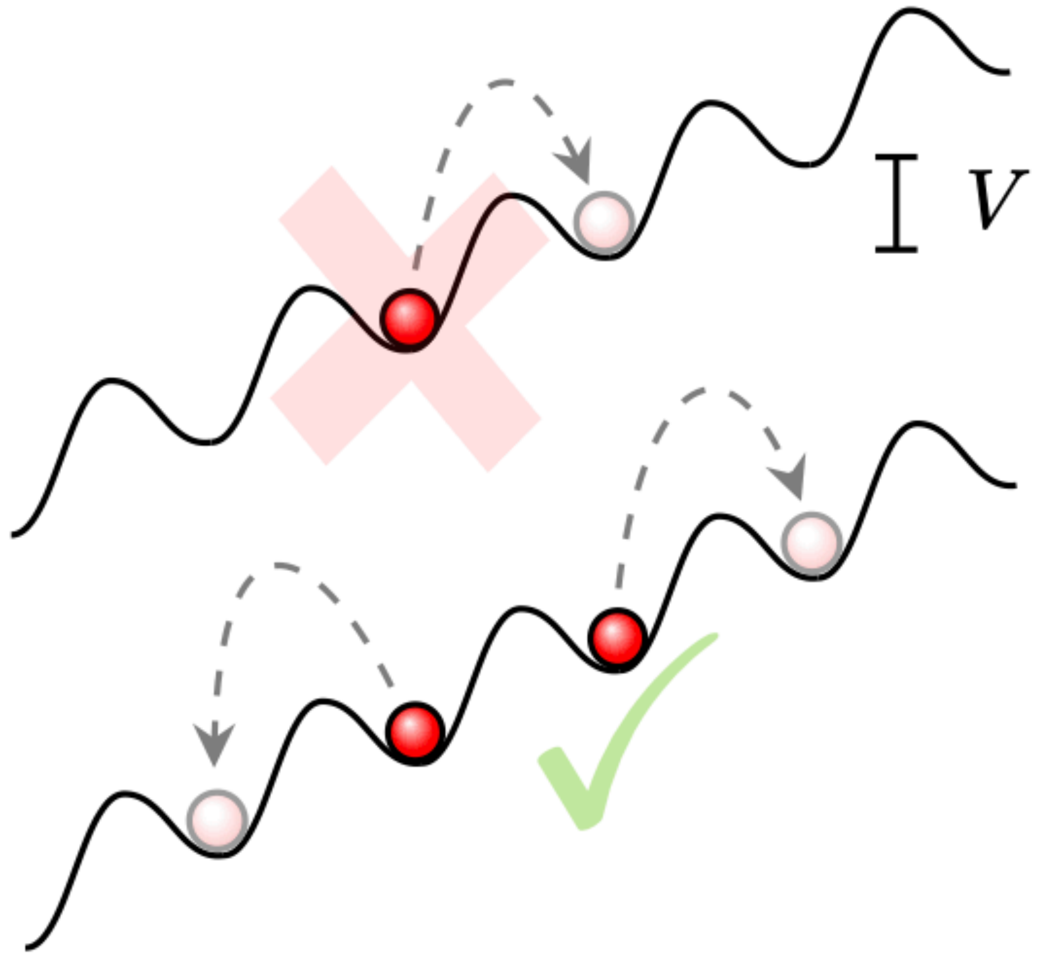
In this talk we explore a class of global symmetries that are modulated in space;

- They present a non-trivial interplay with crystalline symmetries
- Low-energy physics depends on lattice details (UV/IR mixing)

Motivation

Spatially Modulated Symmetries play an important role in systems with constrained dynamics

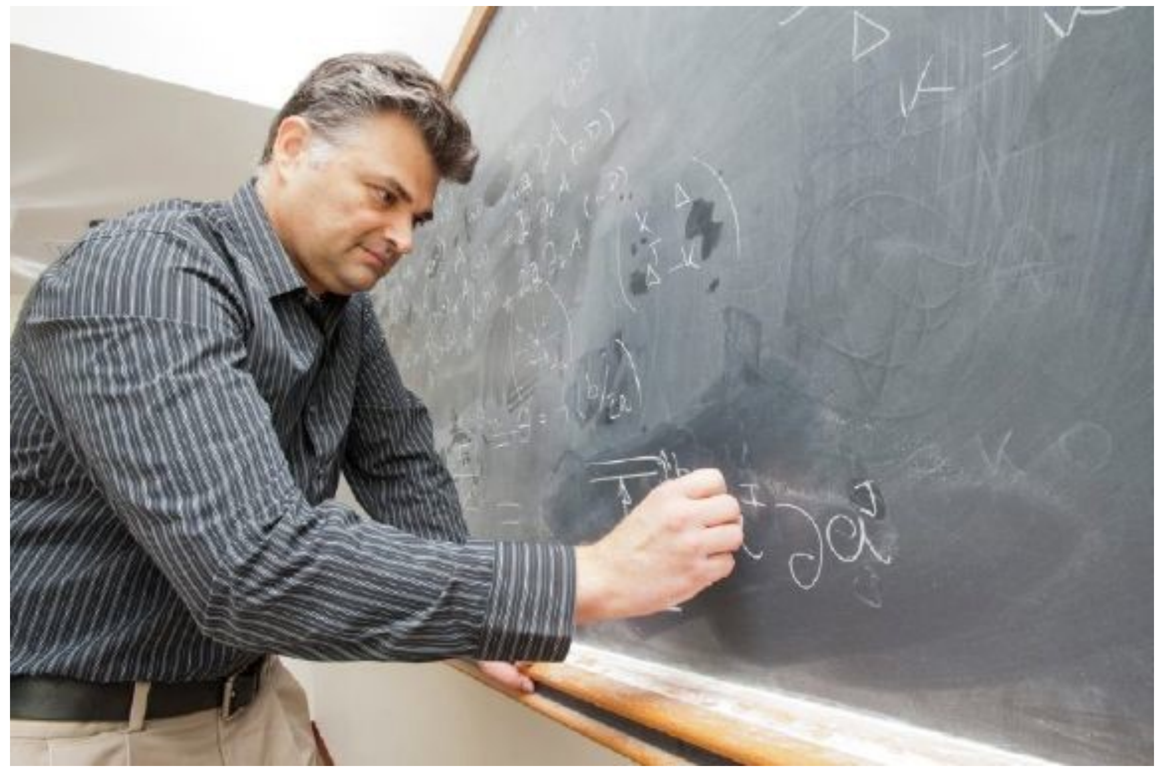
- Fractons (Pretko, M. (2017). PRB, 95(11), 115139)
- Hilbert Space Fragmentation (Khemani, V., Hermele, M., & Nandkishore, R. (2020). PRB, 101(17), 174204)
- Slow Thermalization (Spielman, S. E. et al (2022). arXiv preprint arXiv:2208.02909)
- Tilted lattices (Ethan Lake et al. Phys. Rev. B 107, 195132 (2023))



Plan for the talk

- Overview of discrete symmetries on lattice
- $\mathbb{Z}_N$ Spatially modulated symmetries
- Modulated gauge theories and exotic topological orders

arXiv: 2306.17121, 2310.09490



Claudio Chamon
(BU)

Yizhi You
(NEU)

Omer Aksoy
(MIT)

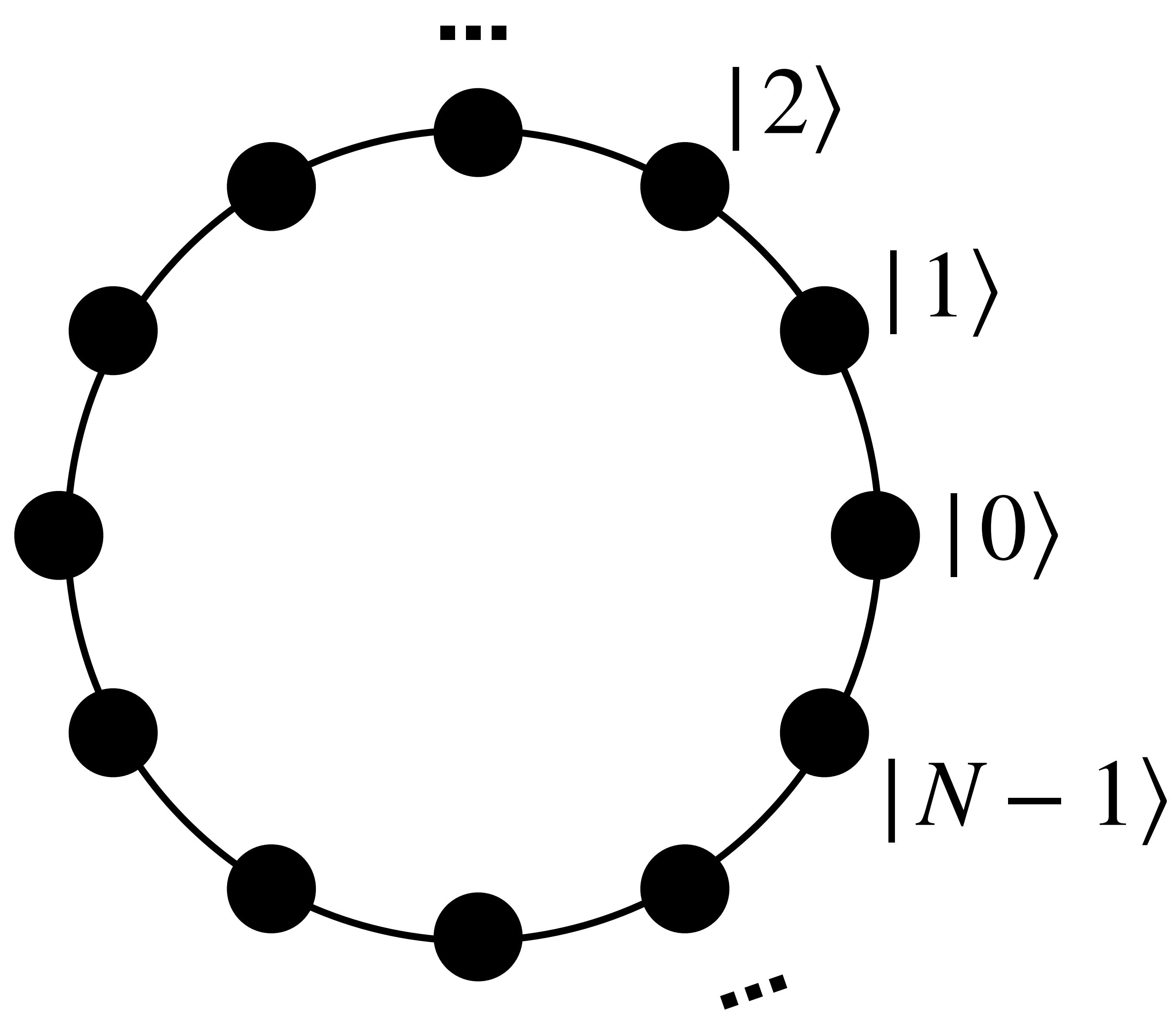
$\mathbb{Z}_N$ "Pauli operators"

- Let us consider a D dimensional lattice Λ with lattice sites $\mathbf{x} \in \Lambda$, withlocal Hilbert space $\mathscr{H}_{\mathbf{x}} \cong \mathbb{C}^N$.
- Local $\mathbb{Z}_N$ unitary operators satisfy the algebra

$$\widehat{X}_{\mathbf{x}} \widehat{Z}_{\mathbf{y}} = e^{\frac{2\pi i}{N}\delta_{\mathbf{x},\mathbf{y}}} \widehat{Z}_{\mathbf{y}} \widehat{X}_{\mathbf{x}}, \quad \widehat{X}_{\mathbf{x}}^N = 1, \quad \widehat{Z}_{\mathbf{x}}^N = 1,$$

- $\widehat{Z}$ and $\hat{X}$ operators both have eigenvalues $\exp\left(\frac{2\pi i}{N}p\right)$ for $p = 0, 1, \ldots, N-1$
- These are known as "clock" and "shift" operators.
  Consider a basis $|p\rangle$ where $\widehat{Z}$ is diagonal: $\widehat{Z}|p\rangle = e^{\frac{2\pi i}{N}p}|p\rangle$. Then $\widehat{X}$ shifts

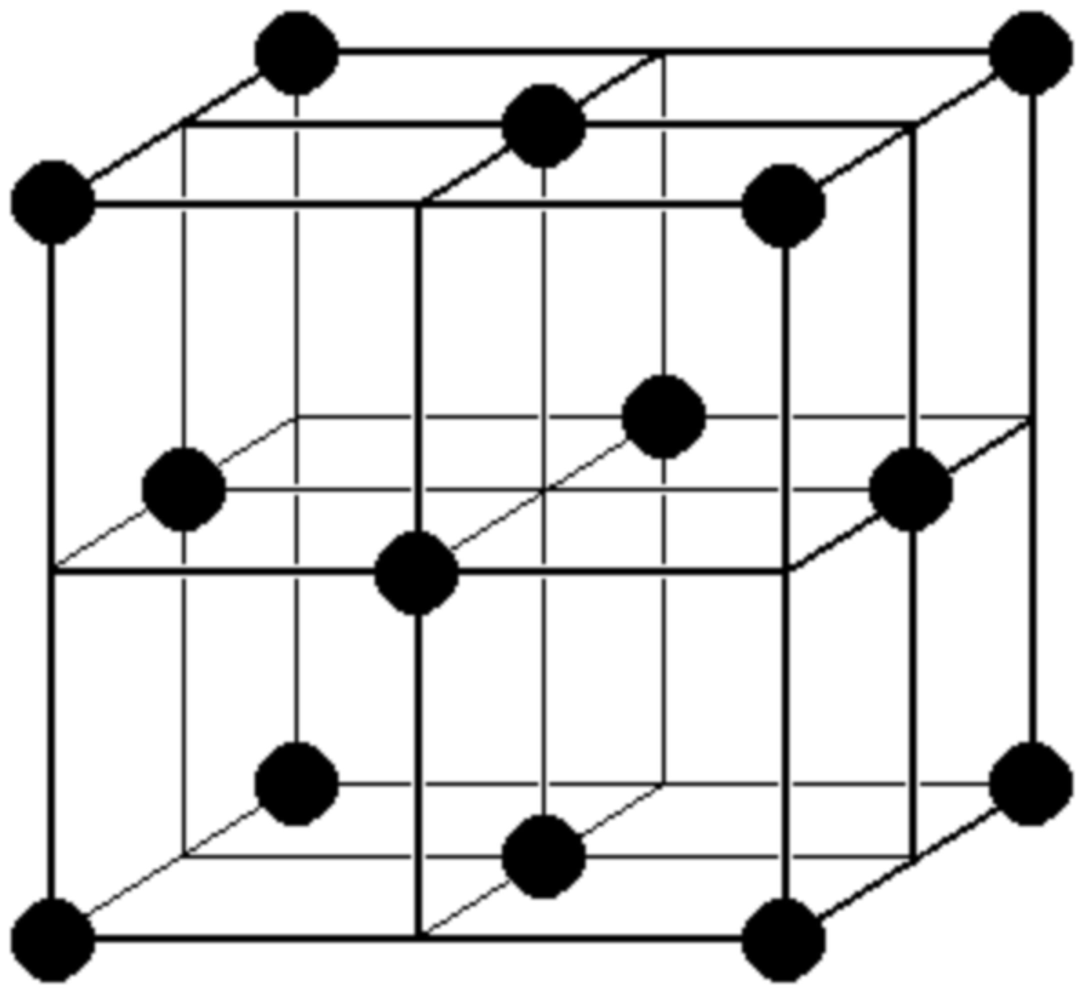
$$\widehat{X}^\dagger |p\rangle = |p+1\rangle$$



- It is also useful to think of them in terms of $\mathbb{Z}_N$ bosons

$$\widehat{X}_{\mathbf{x}} = e^{\mathrm{i}\hat{\theta}_{\mathbf{x}}}, \qquad \widehat{Z}_{\mathbf{x}} = e^{\mathrm{i}\hat{\phi}_{\mathbf{x}}}, \qquad \left[\hat{\theta}_{\mathbf{x}}, \hat{\phi}_{\mathbf{y}}\right] = \mathrm{i}\frac{2\pi}{N}\delta_{\mathbf{x},\mathbf{y}}$$

Symmetry Operators



- Consider the symmetry operator

$$U_\alpha = \prod_{\mathbf{x}\in\Lambda} \left(\widehat{X}_{\mathbf{x}} \right)^{\alpha}$$

or, in terms of generators $U_\alpha = e^{i\alpha G}$, with $G = \sum_{\mathbf{x}\in\Lambda} \hat{\theta}_{\mathbf{x}}$.

- The unitary U_α acts on local operators under conjugation as

$$\widehat{Z}_{\mathbf{x}} \mapsto e^{\frac{2\pi i\alpha}{N}} \widehat{Z}_{\mathbf{x}} \text{ and } \widehat{X}_{\mathbf{x}} \mapsto \widehat{X}_{\mathbf{x}}$$

It obeys $U_\alpha^N = 1$, implementing a $\mathbb{Z}_N$ symmetry indicating that α takes values in $\mathbb{Z}/N\mathbb{Z}$

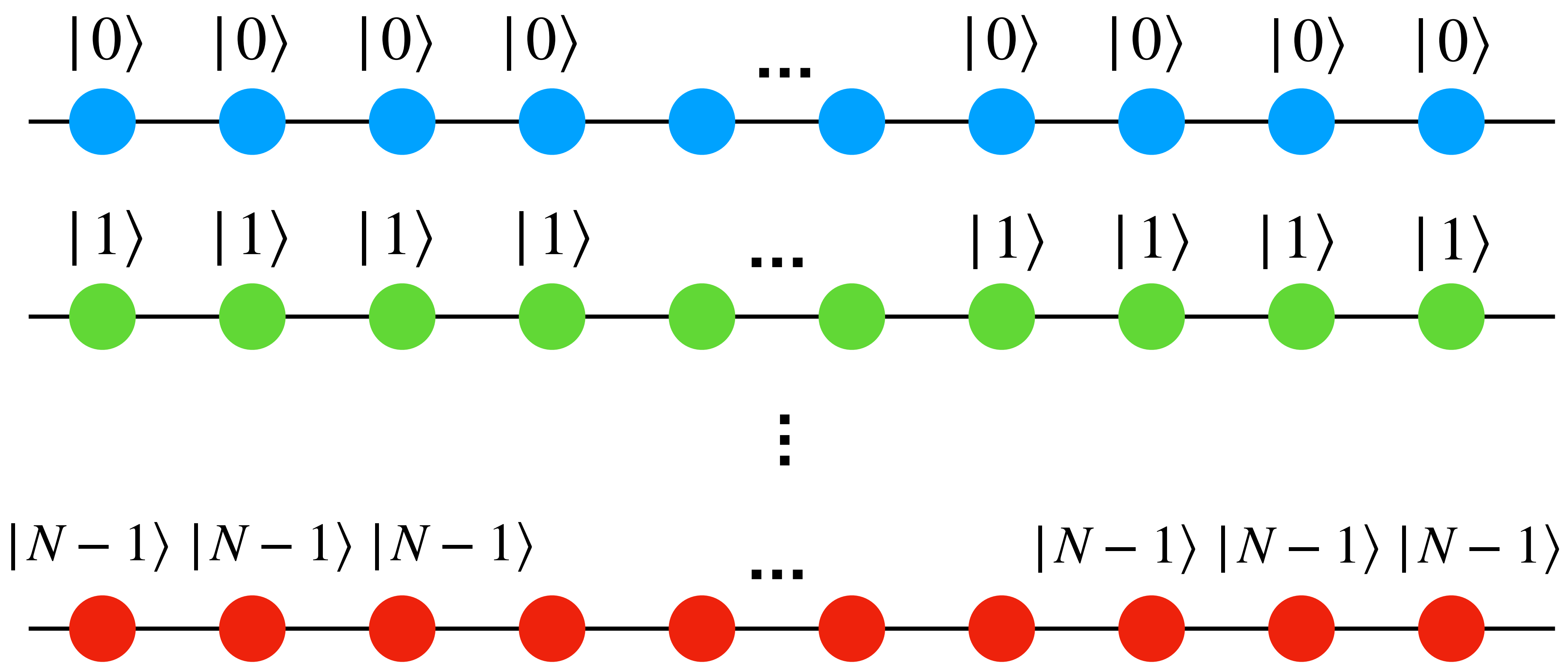
$$\alpha = 0, 1, \ldots, N-1$$

Symmetric Systems

- We can cook up a simple Hamiltonian that is invariant under U_α

$$H = -\sum_{<\mathbf{x},\mathbf{y}>} \hat{Z}_\mathbf{x}\hat{Z}_\mathbf{y}^\dagger - h\sum_\mathbf{x} \widehat{X}_\mathbf{x} + \text{h.c.} + \dots$$

One can explicitly check that $U_\alpha H U_\alpha^\dagger = H$



- In 1D, in the $h \ll 1$ limit, one can understand SSB coming from the N-fold degenerate ground states that satisfy

$$\langle \hat{Z}_\mathbf{x}\hat{Z}_\mathbf{y}^\dagger \rangle = 1$$

- Other sorts of symmetry-related physics can be explored, as SPTs and topological order through gauging ($D \geq 2$)

Spatially Modulated Symmetries

- Consider now the spatially modulated symmetry operator

$$U_\alpha[f] = \prod_{\mathbf{x}\in\Lambda} \left(\widehat{X}_{\mathbf{x}} \right)^{\alpha f_{\mathbf{x}}}$$

for $f_{\mathbf{x}}$ an integer-valued lattice function $f : \Lambda \to \mathbb{Z}$ and $\alpha = 0,1,\ldots,N-1$

- In terms of generators $U_\alpha[f] = e^{i\alpha G[f]}$, with

$$G[f] = \sum_{\mathbf{x}\in\Lambda} f_{\mathbf{x}} \hat{\theta}_{\mathbf{x}}$$

The symmetry acts on local operators as $\widehat{Z}_{\mathbf{x}} \mapsto e^{\frac{2\pi i}{N} f_{\mathbf{x}}} \widehat{Z}_{\mathbf{x}}$ and $\widehat{X}_{\mathbf{x}} \mapsto \widehat{X}_{\mathbf{x}}$

Takeaway: The symmetry acts differently in different points in space

- In general, modulated symmetries do not commute with translations T

$$[T, G[f]] \neq 0$$

Examples of Modulated Symmetries

Examples:

$f_{\mathbf{x}} = 1,$	Charge
$f_{\mathbf{x}} = x, y$	Dipole
$f_{\mathbf{x}} = x^2 - y^2, xy$	Quadrupole
$\vdots$	
$f_{\mathbf{x}} =$ degree n poly	2^n-Multipole
$\vdots$	
$f_{\mathbf{x}} = a^{x+y}, \quad a \in \mathbb{Z}$	Exponentially modulated charge

(Gromov, A. (2019). *Physical Review X*, *9*(3), 031035; Sala, P., Lehmann, J., Rakovszky, T., & Pollmann, F. (2022). *Physical Review Letters*, *129*(17), 170601)

$|\mathbb{Z}_N| = N$ and Periodic Boundaries

The fact that $\widehat{X}_{\mathbf{x}}^{pN} = 1$ for any integer p implies that $U[f]$ acts as identity every time $f_{\mathbf{x}}$ is an integer multiple of N

- This is has an important effect to the mobility of excitations

Periodic boundary conditions

It implies that, in order to have a well defined symmetry operator

$$f_{\mathbf{x}} = f_{\mathbf{x}+L\,\hat{a}} \quad \text{mod } N$$

Which, in general, has the role of reducing the symmetry group. For example, in 1D $\mathbb{Z}_N$ is reduced down to $\mathbb{Z}_q$ with

$$q = \gcd(f_L - f_0, N)$$

The dependence of low-energy observables (e.g. $\mathbb{Z}_q$ symmetry defects) on the lattice size L (a UV regularization) is a common feature in modulated systems, and is known as UV/IR mixing.

Modulated vs Crystalline Symmetries

- There is a non-trivial interplay between lattice (crystalline) and modulated symmetries.
- One can see, even classically, that dipole moment and charge mix under translations

$$T(\mathbf{a}) : \sum_{\mathbf{x}} q_{\mathbf{x}} \mathbf{x} \to \sum_{\mathbf{x}} q_{\mathbf{x}} (\mathbf{x} + \mathbf{a})$$

$$= \underbrace{\sum_{\mathbf{x}} q_{\mathbf{x}} \mathbf{x}}_{\text{dipole moment}} + \mathbf{a} \underbrace{\sum_{\mathbf{x}} q_{\mathbf{x}}}_{\text{charge}}$$

which also manifests in the quantum theory

Modulated vs Crystalline Symmetries

Under n sites translation $T_a^{(n)}$ along the a-th direction, it is possible to show that under conjugation

$$T_a^{(n)} U[f] T_a^{(n)\dagger} = U[f] U[\Delta_a^{(n)} f],$$

where $\Delta_a^{(n)}$ is the lattice n-th order derivative of f along the a-th direction.

Example: For $n = 2$ $\Rightarrow$ $\Delta_a^{(2)} = f(\mathbf{x} - \hat{a}) - 2f(\mathbf{x}) + f(\mathbf{x} + \hat{a})$

- Roughly, translation invariance requires that if f is degree q polynomial, one must also take into account all polynomials with degree less than q.

- Systems with closed symmetry algebra must be symmetric under a combination of generators

$$G[f_1] \oplus \ldots \oplus G[f_q]$$

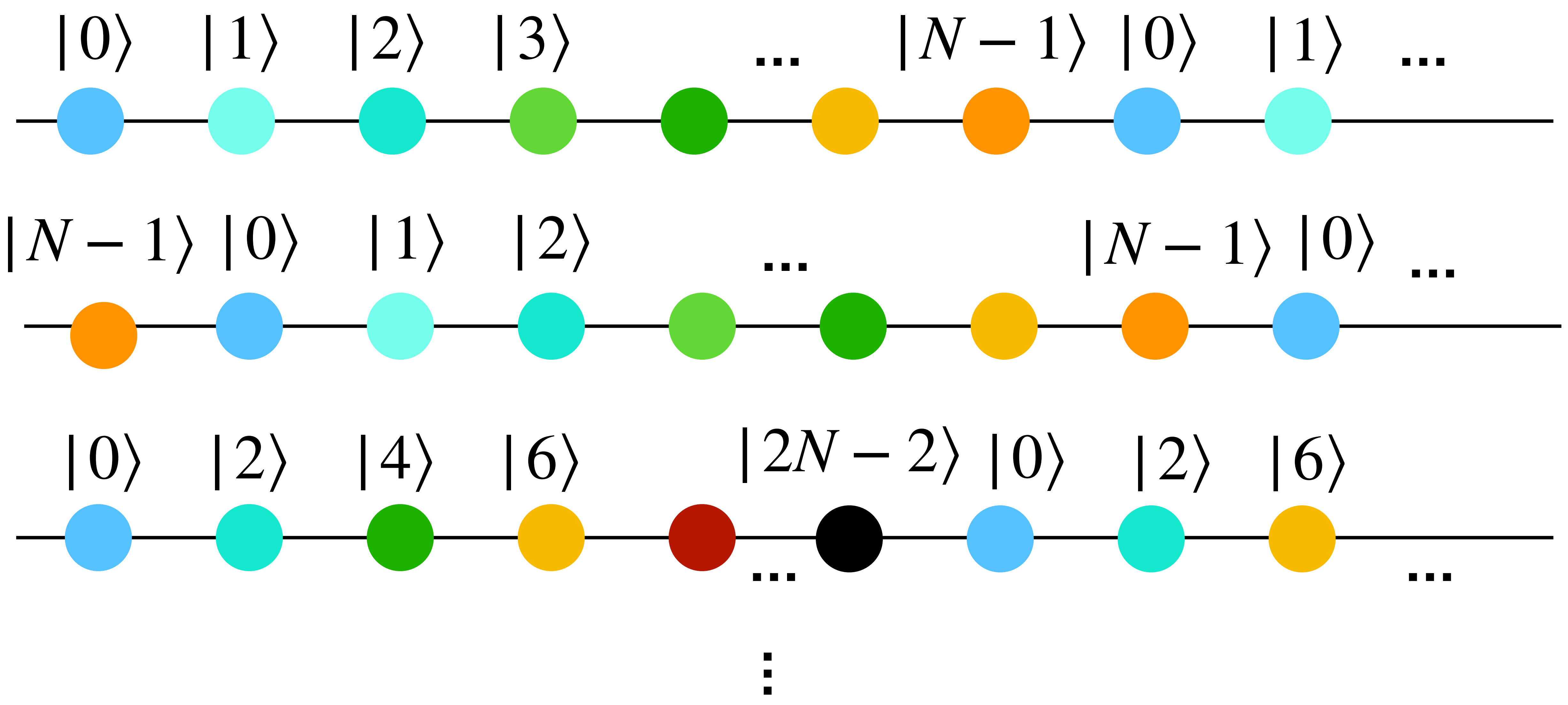
It is possible to also take into account other constraints by requiring other crystalline symmetries (e.g. rotations) in generic lattices (Bulmash, D., Hart, O., & Nandkishore, R. (2023). *SciPost Physics*, *15*(6), 235.)

Dipole Symmetry SSB

- As an example, let us consider the lattice Hamiltonian that conserves both dipole and charge $G[1] \oplus G[x_1] \oplus \ldots \oplus G[x_d]$

$$H = -\sum_{\mathbf{x},a} \widehat{Z}_{\mathbf{x}-\hat{a}} \widehat{Z}^{\dagger\,2}_{\mathbf{x}} \widehat{Z}_{\mathbf{x}+\hat{a}} - h \sum_{\mathbf{x}} \widehat{X}_{\mathbf{x}} + \text{h.c.} + \ldots$$

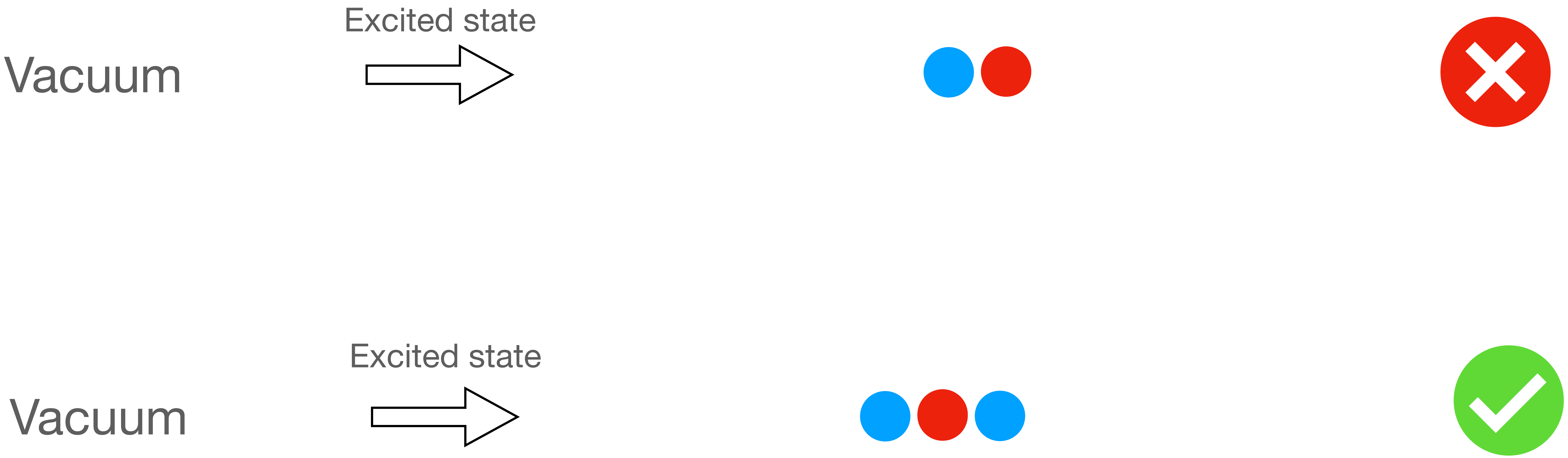
$h \ll 1$ **limit in 1D:**



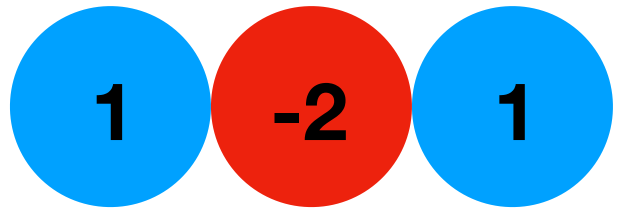
Spatially modulated degenerate ground states (Lake, E., Hermele, M., & Senthil, T. (2022). *Physical Review B*, *106*(6), 064511)

Domain Walls Dipole Conserving Dynamics

- Let us denote excitations with positive/negative charge $q_{\mathbf{x}}$ by blue/red dots.



Dipole conservation mod N

- The fact that charge and dipole moment of excitations are defined only mod N allow us to stack excitations 

on the top of each other until they disappear back to the vacuum

which is implemented by the modulated operator $\prod_{i=1}^{N} \widehat{X}^{i}_{\mathbf{x}+i\hat{a}}$

Beyond SSB

SPTs: One can also study Symmetry Protected Topological ordered phases associated to modulated symmetries (Han, J. H., Lake, E., Lam, H. T., Verresen, R., & You, Y. (2023). *arXiv:2309.10036;* Lam, H. T. (2023). *arXiv:2311.04962*)

Topological Order: Gauging modulated symmetries is a mechanism to get topologically ordered with locally constrained anyons

Deconfined phase of gauge theory $\leftrightarrow$ Topological Order

Gauging $U_\alpha[f]$

In general, the range of interaction depends on the function f_r we choose. Schematically:

$$H_{\text{matter}} = -t \sum_{\mathbf{x},a} \prod_i \hat{Z}_{\mathbf{x}+i\hat{a}}^{\Delta_a^i} + \text{h.c.} + \ldots$$

where the Δ_a^i are coefficients that define a “lattice derivative” operator

$$\sum_i \Delta_a^i f_{\mathbf{x}+i\hat{a}} \equiv \Delta_a f$$

The coefficients Δ_a^i are determined by the condition $\Delta_a f = 0$.

Symmetry	$f_{\mathbf{x}}$	Δ_a	$\prod_i \hat{Z}_{\mathbf{x}+i\hat{a}}^{\Delta_a^i}$
Charge	1	$f_{\mathbf{x}+\hat{a}} - f_{\mathbf{x}}$	$\hat{Z}_{\mathbf{x}+\hat{a}}\hat{Z}_{\mathbf{x}}^{\dagger}$
Dipole	x, y	$f_{\mathbf{x}+\hat{a}} - 2f_{\mathbf{x}} + f_{\mathbf{x}-\hat{a}}$	$\hat{Z}_{\mathbf{x}+\hat{a}}\hat{Z}_{\mathbf{x}}^{\dagger\,2}\hat{Z}_{\mathbf{x}-\hat{a}}$
⋮			
Exponential	m^{x+y}, for $m \in \mathbb{N}$	$f_{\mathbf{x}+\hat{a}} - m f_{\mathbf{x}}$	$\hat{Z}_{\mathbf{x}+\hat{a}}\hat{Z}_{\mathbf{x}}^{\dagger\,m}$

(Delfino, G., Chamon, C., & You, Y. (2023). *arXiv:2306.17121*)

Gauging $U_\alpha[f]$

- We now require local invariance under

$$Z_{\mathbf{x}} \to e^{i\alpha_{\mathbf{x}}} Z_{\mathbf{x}},$$

where $\alpha_{\mathbf{x}}$ now depends on the lattice position $\mathbf{x} \in \Lambda$.

- As usual, in order to save invariance under $\alpha_{\mathbf{x}}$, we introduce gauge fields A_a and E_a that minimally couple with the matter fields

$$H_{\text{matter+gauge}} = -\sum_{\mathbf{x},a} e^{-iA_{a,\mathbf{x}}} \prod_i \widehat{Z}^{\Delta_a^i}_{\mathbf{x}+i\hat{a}} + \text{h.c.} + \dots$$

Under gauge transformations, both the matter and gauge degrees of freedom transform

$$Z_{\mathbf{x}} \to e^{i\alpha_{\mathbf{x}}} Z_{\mathbf{x}} \qquad \text{and} \qquad A_{a,\mathbf{x}} \to A_{a,\mathbf{x}} + \Delta_a \alpha_{\mathbf{x}}$$

2D Modulated Gauge Theory

- From the canonical relation $[A_{a,\mathbf{x}}, E_{b,\mathbf{y}}] = i\delta_{a,b}\delta_{\mathbf{x},\mathbf{y}}$, gauge transformations define a Gauss Law

$$q_{\mathbf{x}} = \Delta_a E_a \,.$$

- One can also define a gauge invariant magnetic flux, schematically expressed as

$$b_{\mathbf{x}} = \check{\Delta}_a A_a$$

where $\check{\Delta}_a$ is defined such that $\check{\Delta}_a \Delta_a \alpha = 0$ for any lattice function $\alpha_{\mathbf{x}}$, which enforces that $b_{\mathbf{x}}$ is gauge invariant.

- The specifics of both Δ_a and $\check{\Delta}_a$depend on the function $f_{\mathbf{x}}$ and the lattice symmetries one wishes to preserve.

Modulated Topological Order

- This procedure provide us exactly solvable two-dimensional lattice realizations of $G[f_1] \oplus \ldots \oplus G[f_n]$ gauge theories

Examples:

$$\mathbb{Z}_N \text{ Toric Code} \leftrightarrow G[1]$$

(Kitaev, A. Yu. Annals of Physics 303.1 (2003): 2-30)

$$\mathbb{Z}_N \text{ Rank 2 Toric Code} \leftrightarrow G[1] \oplus G[x] \oplus G[y]$$

(Oh, Y. T., Kim, J., Moon, E. G., & Han, J. H. (2022). Physical Review B, 105(4), 045128)

Modulated Topological Order in 2D

- Anyons have restricted mobility, now incorporated in the local Gauss law and gauge invariant magnetic flux

- Ground state degeneracy depends on system size

- Anyons have position-dependent quantum numbers (Pace, S. D., & Wen, X. G. (2022). *Physical Review B*, *106*(4), 045145)

- For $\mathcal{O}(1)$ generators $G[f_1] \oplus \ldots \oplus G[f_n]$, low energy effective theories are equivalent to twisted TQFT gauge theories

- Different modulated topologically ordered phases can be connected through phase transitions driven by **anyon condensations**

Phase Transitions

Anyon condensation provide us a direct mechanism to relate different topological orders
(Delfino, G., & You, Y. (2023). *arXiv:2310.09490)*

$G[1] \oplus \overline{G}[1]$
Double $\mathbb{Z}_N$ Top. Order

$\longrightarrow$

$G[1] \oplus G[x] \oplus G[y]$
PRB 105.4 (2022): 045128

$G[1] \oplus G[x] \oplus G[y] \oplus G[xy]$
SciPost Physics 14.1 (2023): 002

$G[1] \oplus G[x] \oplus G[y] \oplus G[x^2 + y^2]$
PRL 127.10 (2021): 107202

Some cool results:

- Anyon condensation can cause the emergence of extra (modulated) symmetries

- As one condenses some of the anyons, even more can emerge $\mathscr{D}_{\text{before}} < \mathscr{D}_{\text{after}}$

Equivalence to Subsystem Symmetries

- Fracton topological order is understood in terms of subsystem symmetries
  (Vijay, S., Haah, J., & Fu, L. (2016). Physical Review B, 94(23), 235157)

- Consider a subsystem symmetry operator $U_\Omega^\alpha = e^{i\alpha Q_\Omega}$ with support on a sub-lattice $\Omega \subset \Lambda$ of the whole lattice Λ

(Subsystem) $$Q_\Omega = \sum_{\mathbf{x}\in\Omega} q_\mathbf{x} = \sum_{\mathbf{x}\in\Lambda} \delta_\mathbf{x}(\Omega) q_\mathbf{x}$$ (Modulated)

With this, usual 3D fracton systems can be casted as a modulated gauge theories associated to $\oplus_i^{\mathcal{O}(L)} G[\delta(\Omega_i)]$, for an extensive number of sub-lattices $\Omega_i \subset \Lambda$.

Final Considerations

- Natural way of incorporating **constrained mobility** excitations locally;
- **Sensitivity to geometry**: UV/IR mixing, position-dependent anyons, etc;
- Classification and characterization of quantum phases are mapped into the study of **functions and number theory**;
- Modulated gauge theories can take into account a **whole spectrum of exotic Abelian phases**

$G[1]$ $G[\text{poly}]$ $G[\text{non-poly}]$ $\oplus_i G[\delta(\Omega_i)]$ $\cdots$ $\longrightarrow$

Topological order Fractons ?

Thank you!