U.S. DEPARTMENT OF ENERGY
Office of Science

BOSTON UNIVERSITY

Gauging Modulated Symmetries in 1+1D

Gui Delfino

arxiv: 2406.12962

Sal Pace

Ho Tat Lam

$\in$ MIT

Omer Aksoy

Introduction

- Typically, internal and space symmetry groups G_{int} and G_{space} do not talk to each other
- Spatially **modulated symmetries**, on the other hand, act non-uniformly in space and have a non-trivial interplay with space-like symmetries
- This is incorporated through a **group homomorphism** which defines a non-trivial action $\varphi : G_{\text{space}} \to \text{Aut}(G_{\text{int}})$
- Total symmetry group $G = G_{\text{int}} \rtimes_{\varphi} G_{\text{space}}$
- Here we will be mostly interested in the case where G_{int} is composed of Abelian 0-form symmetries, but generalizations are straightforward

arxiv: 2406.12962

Dipole Moment

Consider $U(1) \times U(1)^d$ symmetry generated by **dipole moment** and **charge** in (d+1) dimensions

$$U_q = \exp\left(\mathrm{i}\,\alpha_0\, Q\right), \quad \text{and} \quad U^{(i)}_{dip} = \exp\left(\mathrm{i}\,\alpha_i\, D_i\right) \quad \text{with} \quad Q = \int d\mathbf{x}\, \rho(\mathbf{x}) \quad \text{and} \quad D_i = \int d\mathbf{x}\, \mathrm{x}_i\, \rho(\mathbf{x})$$

- Translations $T_{\mathbf{a}} : \mathbf{x} \to \mathbf{x} + \mathbf{a}$ have a non-trivial action $T_{\mathbf{a}}\, U^{(i)}_{dip}\, T^{\dagger}_{\mathbf{a}} = U^{(i)}_{dip}\, U_q^{\mathrm{a}_i}$

Total symmetry group $G = (U(1) \times U(1)^d) \rtimes_{\varphi} G_{\text{space}}$

$\Rightarrow$ More generally, the modulation can be implemented by **arbitrary functions**

$$G^{(q)} = \int d\mathbf{x}\, f^{(q)}(\mathbf{x})\, \rho(\mathbf{x}), \quad q = 1, \ldots, n, \quad \text{with} \quad f^{(q)} : \text{space} \to \mathbb{R}$$

arxiv: 2406.12962

General Remaks

⇒ One must be careful whether $G^{(q)}$ are well defined quantities

⇒ Modulated symmetries gives rise to **UV/IR mixing**, which challenges separation of scales principle;

~For gapless phases, there are zero energy modes with **arbitrarily high momentum** [Seiberg, Shao 2021]

~For gapped phases, the **ground state degeneracy** depend on the lattice regularization of underlying space manifold

arxiv: 2406.12962

Fruitful Idea

- In 2+1d, it's a general mechanism to get non-trivially symmetry **enriched topological orders** (SETs) [Pace, Wen 2022; GD et al 2023]

- In 3+1d, gauging dipole symmetry provides effective field theory descriptions of **fracton** systems [Pretko, 2018]

⇒ Can we have new physics by gauging **arbitrary modulated symmetries**?

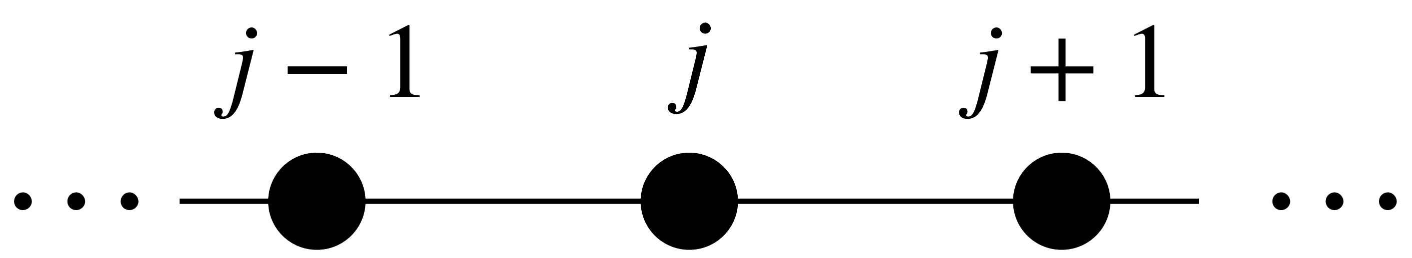
~Gauging arbitrary functions in the continuum in higher dimensions, however, can be a rather complicated task

⇒ Alternatively, it is easier to gauge **discrete symmetries in the lattice**, as opposite to the continuum

arxiv: 2406.12962

Lattice Discrete QFTs in 1+1D

- Let us consider **one-dimensional lattice** chains Λ



- On each site there is a $\mathbb{Z}_N$ **degree of freedom** (*qudit*)

$$Z_j X_k = e^{\frac{2\pi \mathrm{i}}{N}\delta_{jk}} X_k Z_j \qquad \text{and} \qquad X_j^N = Z_j^N = 1$$

- We consider **finite Abelian groups**, which constrains the functions in symmetry operators

$$\boxed{U_f = \prod_j X_j^{f_j}} \qquad \text{to be} \qquad f : \Lambda \to \mathbb{Z}_N$$

arxiv: 2406.12962

Warm up

- Let us consider the **dipole** conserving Hamiltonian on infinite chain

$$H(g) = -\sum_j Z_{j-1} Z_j^{\dagger\,2} Z_{j+1} - g \sum_j X_j + \text{h.c.}$$

- Internal symmetry group $G_{int} = \mathbb{Z}_N \times \mathbb{Z}_N$

$$Q = \prod_j X_j \quad \text{(“charge”)} \qquad\qquad D = \prod_j X_j^j \quad \text{(“dipole”)}$$

- Translations $T : j \to j+1$ implies the homomorphism $T\,D\,T^\dagger = D\,Q$

- Total symmetry group $G = G_{int} \rtimes_\varphi \mathbb{Z}$

arxiv: 2406.12962

Peak into UV/IR mixing

- Under periodic boundary conditions $j \sim j + L$, and $T^L = 1$. The modulation implies that

$$D = Q^L D$$

- In order to have a **well defined** symmetry operator, one must instead consider the rescaled operator $D^m = Q^{mL} D^m$ such that $m L = 0 \mod N$. Thus

$$\tilde{D} = D^m = \prod_j X_j^{\frac{N}{\gcd(N,L)} j}$$

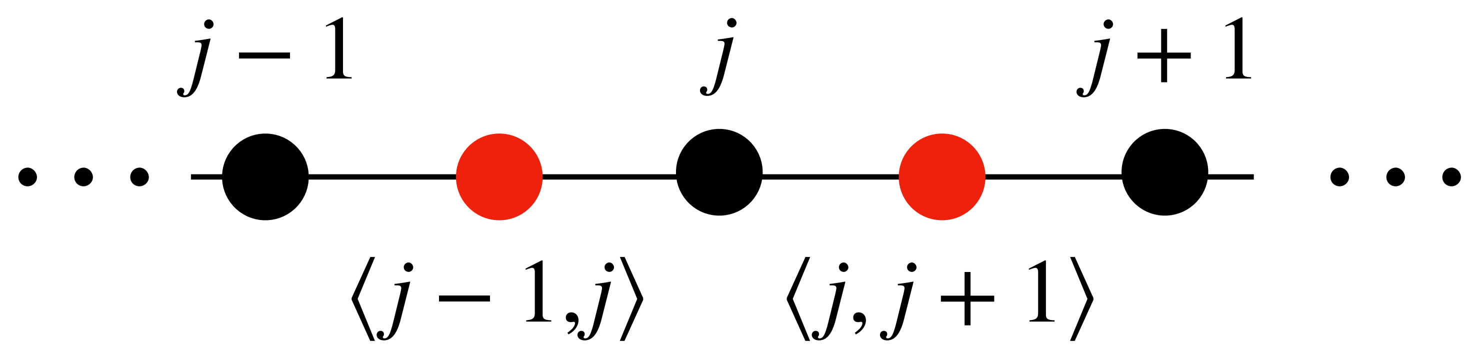
is a proper generator **compatible with boundary conditions**. Here $\gcd$ stands for greatest common divisor.

- The operators Q and $\tilde{D}$ no longer generate $\mathbb{Z}_N \times \mathbb{Z}_N$ group, but instead $\mathbb{Z}_N \times \mathbb{Z}_{\gcd(N,L)}$

~ This impacts the **IR physics** of the SSB phase (e.g. ground state degeneracy) [Gorantla at al, 2022]

arxiv: 2406.12962

Gauging Dipole Symmetry



- We then proceed to **gauge** the $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetries by

  1. Extending the total Hilbert space by adding new $\mathbb{Z}_N$ d.o.f on links $(\tilde{X}_{j,j+1}, \tilde{Z}_{j,j+1})$
  2. Imposing the Gauss law $\quad G_j = X_j \tilde{Z}^{\dagger}_{j-2,j-1} \tilde{Z}^{2}_{j-1,j} \tilde{Z}^{\dagger}_{j,j+1} = 1$

$\Rightarrow$ **Dual dipole theory** (after gauge fixing)

$$H^{\vee}(g) = -g \sum_j \tilde{Z}_{j-2,j-1} \tilde{Z}^{\dagger\,2}_{j-1,j} \tilde{Z}_{j,j+1} - \sum_j \tilde{X}_{j-1,j} + \text{h.c.}$$

with dual symmetries

$$Q^{\vee} = \prod_j \tilde{X}_{j,j+1}\,; \qquad D^{\vee} = \prod_j \tilde{X}^{j}_{j,j+1}$$

arxiv: 2406.12962

KW-like duality

- **Self-duality:** under gauging, the resulting theory looks the same as the original one
- Whenever this happens, **Kramers-Wannier** (KW) dualities hold

$$H^{\vee}(g) = g\,H(g^{-1})$$

- Operators which implement this duality correspond to **non-invertible symmetries** at $g = 1$ [Seiberg, Shao, 2024]

  ~Not described by group structure, but fusion algebra instead

- The presence of non-invertible symmetry may forbid trivially gapped ground states

arxiv: 2406.12962

Spatially Modulated Symmetries

- Consider the symmetry operators (for $q = 1,\ldots,n$)

$$U_q = \prod_j X_j^{f_j^{(q)}} \text{ and arbitrary functions } f^{(q)} : \Lambda \to \mathbb{Z}_N$$

- Translation symmetry requires the set of functions $S = \{f^{(q)}\}_{q=1,\ldots,n}$ must be **closed under translations**

- A local Hamiltonian is then built out of **symmetric terms**

$$H(\Delta; g) = -\sum_j \prod_\ell Z_\ell^{\Delta_{j\ell}} - g\sum_j X_j + \text{h.c.}$$

where $\mathbb{Z}_N$ **coefficients** $\Delta_{j\ell}$ must obey

$$\sum_\ell \Delta_{j\ell} f_\ell^{(q)} = 0 \mod N \quad \forall j \in \Lambda.$$

arxiv: 2406.12962

Examples

$\Rightarrow$ The Δ "matrix" is rather sparse due to (small) **interaction range** r, i.e., $\Delta_{j\ell} = 0$ if $|j - \ell| > r$

- **Dipole symmetry** $f_j = j \quad \Leftrightarrow \quad \Delta_{j\ell} = \delta_{j,\ell-1} - 2\delta_{j,\ell} + \delta_{j,\ell+1}$

$$(\Delta \cdot h)_j = h_{j+1} - 2h_j + h_{j-1}$$

- **Polynomial degree m symmetries** $f_j = j^m \quad \Leftrightarrow \quad \Delta_{j\ell} = [\partial^{m+1}]_{j,\ell}$

- **Exponential symmetries** $f_j = a^j \quad \Leftrightarrow \quad \Delta_{j\ell} = a\delta_{j,\ell} - \delta_{j,\ell+1}$

$$(\Delta \cdot h)_j = h_j - a\, h_{j-1}$$

for $a \in \mathbb{Z}/N\mathbb{Z}$ and $\gcd(a, N) = 1$.

arxiv: 2406.12962

Gauging modulated symmetries

$$G = G_{\text{int}} \rtimes_{\varphi} G_{\text{space}} \overset{gauging\ G_{int}}{\Rightarrow} G = G_{\text{int}} \rtimes_{\varphi^\vee} G_{\text{space}}$$

- In general, the symmetry can have different modulation φ and $\varphi^\vee$

Question: When are theories **self-dual** under gauging?

~For N **prime** and set of functions $S = \{f^{(q)}\}_{q=1,\ldots,n}$ closed under translations

~In constrast, using technique of ring theory and regular matrices, we show that for **non-prime** N there might be no KW-like duality.

arxiv: 2406.12962

KW and non-invertible symmetries

- Under gauging a modulated symmetry, the **Kramers-Wannier duality** reads

$$\boxed{H^\vee(\Delta; g) = g\, H(\Delta^T; g^{-1})} \quad \text{where} \quad \Delta^T_{j\ell} = \Delta_{\ell j};$$

- These are naturally interpreted as **non-invertible reflections,** which are more fundamental than KW symmetry itself

- We explicitly construct the **non-invertible KW duality operators** and derive their properties and fusion rules

$$\mathrm{D}_{KW}\mathrm{D}_{KW} = \mathfrak{H}^{\sigma+1}\, T^{(n \mod 2)} \prod_{q=1}^{n} \sum_{a=0}^{N-1} U_q^a, \qquad \mathrm{D}_{KW} U_q = U_q \mathrm{D}_{KW} = \mathrm{D}_{KW}$$

arxiv: 2406.12962

Future Perspectives

- Flow into **continuum** of modulated symmetries;
- Self dual transition **critical points**;
- Higher dimensions and **modulated higher-form symmetries**;
- $\vdots$
- **SymTFT** realizations (see Sal's seminar on Friday!)

arxiv: 2406.12962

Thank you!

arxiv: 2406.12962