TOPOLOGICAL META-MATERIALS FROM COOPER-PAIR SPLITTERS

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PRX Quantum 6, 030318 (2025)

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Introduction

- Topological phases of matter are gapped and highly entangled (spin liquids, topological superconductors, etc.)
- They are hard to realize experimentally!
- Alternative: Artificial "**meta-materials**"
- Our approach: networks of usual **superconducting wires** under magnetic fields

$\Rightarrow$ These networks are built out of smaller blocks: **Cooper-pair splitters**

We find a phase diagram with topological superconductivity

arXiv: 2408.06420

Topological Superconductors

- Ten-fold classification: classifies free-fermion gapped Hamiltonians based on the presence of symmetries (Altland & Zirnbauer, 1997)
- Topological superconductors preserve particle-hole symmetry (classes C and D)
- This table suggests that $d = 2$ appears to be the natural dimension to work on;
- Additionally, we must **break time reversal symmetry**

AZ	Symmetry T	C	S	Dimension 1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

- T: time-reversal,
- C: particle-hole
- S: chiral S = TC

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Chern numbers in 2D

- The topological invariant in two-dimensions correspond to **Chern numbers**
- Translation invariant Hamiltonians eigenstates $|n\mathbf{k}\rangle$ can be labelled by band index n and crystal momentum $\mathbf{k}$
- The Chern numbers for the n-th band can be computed through the sum over the Brillouin zone

$$c_n = \frac{1}{2\pi} \sum_{\mathbf{k}\in BZ} \nabla_{\mathbf{k}} \times \overrightarrow{A}_{n,\mathbf{k}}$$

with $\overrightarrow{A}_{n,\mathbf{k}} = -i\langle n\mathbf{k}|\overrightarrow{\nabla}_{\mathbf{k}}|n\mathbf{k}\rangle$ is the Berry connection in the n-th Brillouin zone

- Chern numbers are **quantized** $c_n \in \mathbb{Z}$, constrained to $\sum_n c_n = 0$

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Example: Chern insulator

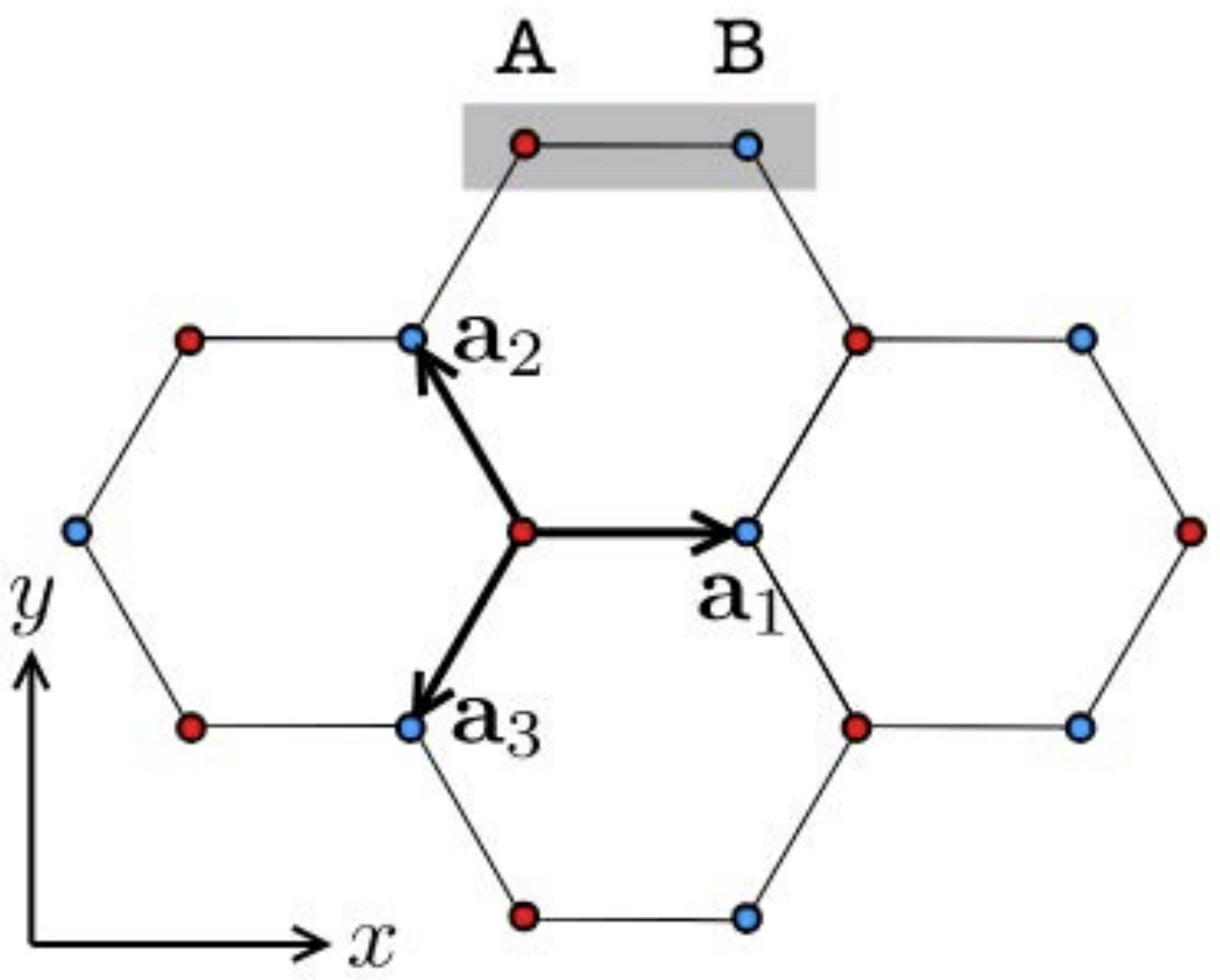
- Chern insulators belong to class A: respect neither T, C, or S, and in 2D are classified by $\mathbb{Z}$

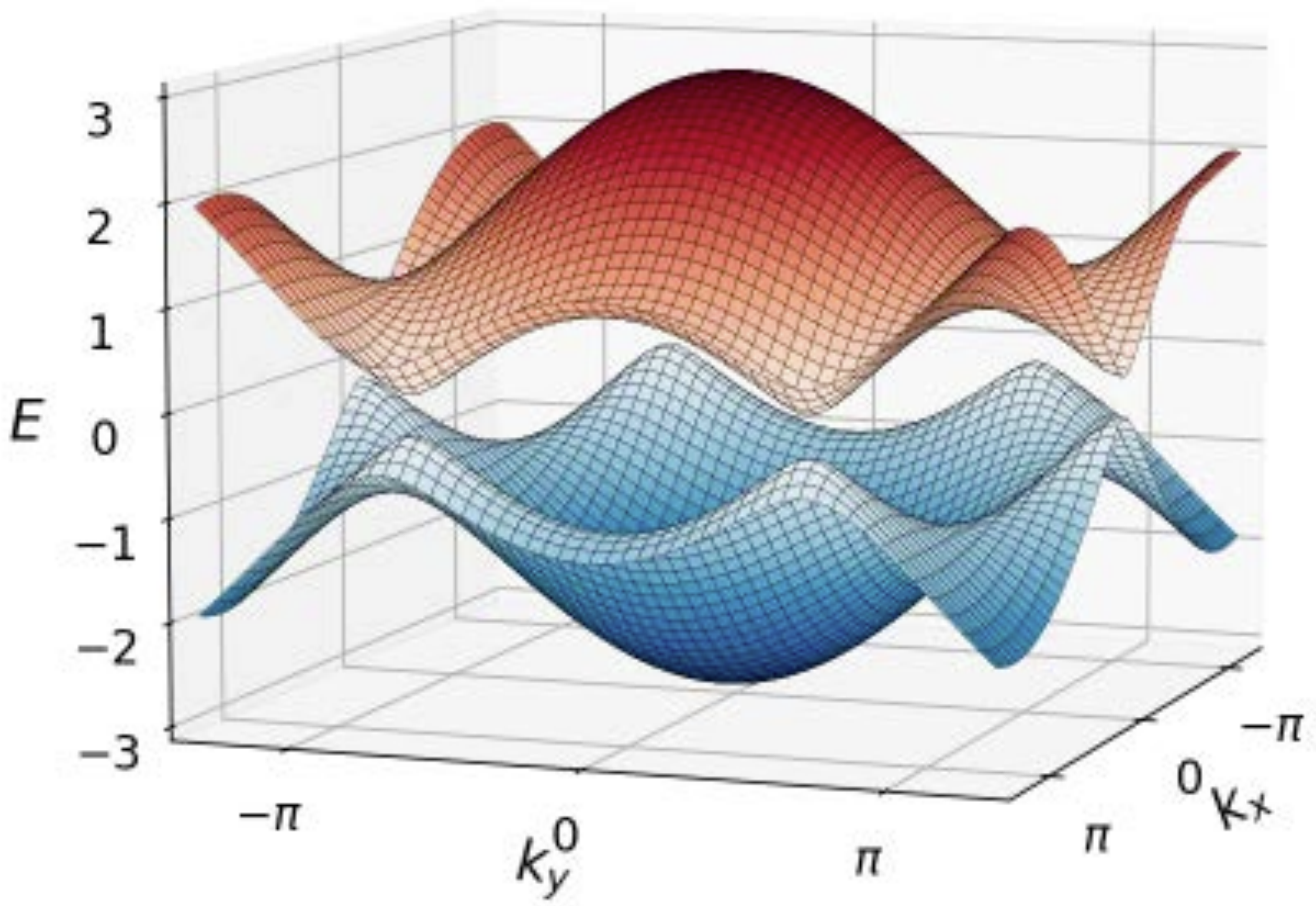
- Haldane model (Haldane, 1988)

$$H = t_1 \sum_{\langle i,j \rangle} c_i^\dagger c_j + t_2 \sum_{\langle\langle i,j \rangle\rangle} e^{\pm i\varphi} c_i^\dagger c_j + m \sum_i \epsilon_i c_i^\dagger c_i,$$

- Hopping electrons in a magnetic field $\varphi = \pi/2$

- In general, non-trivial φ (mod 2π) breaks time-reversal symmetry





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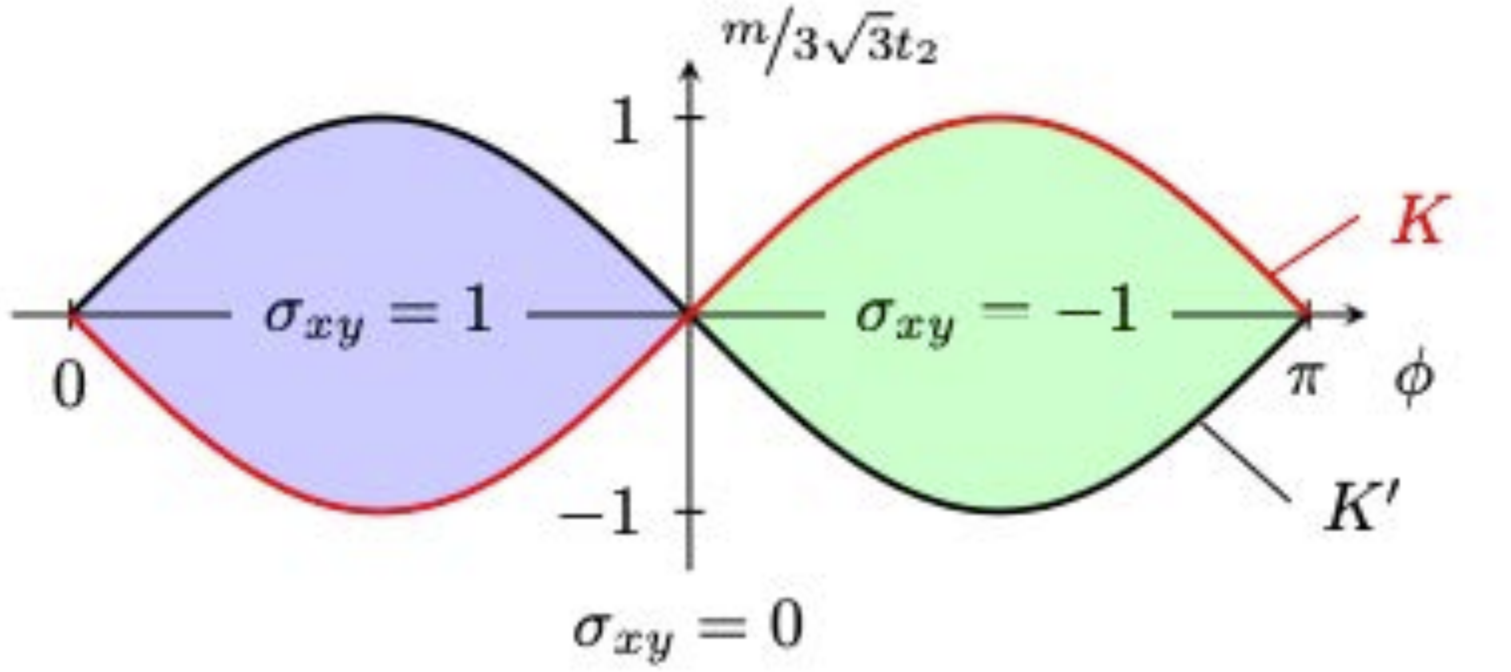
Chern numbers

- There are three gapped phases which we can access by varying t_1, t_2, m and φ

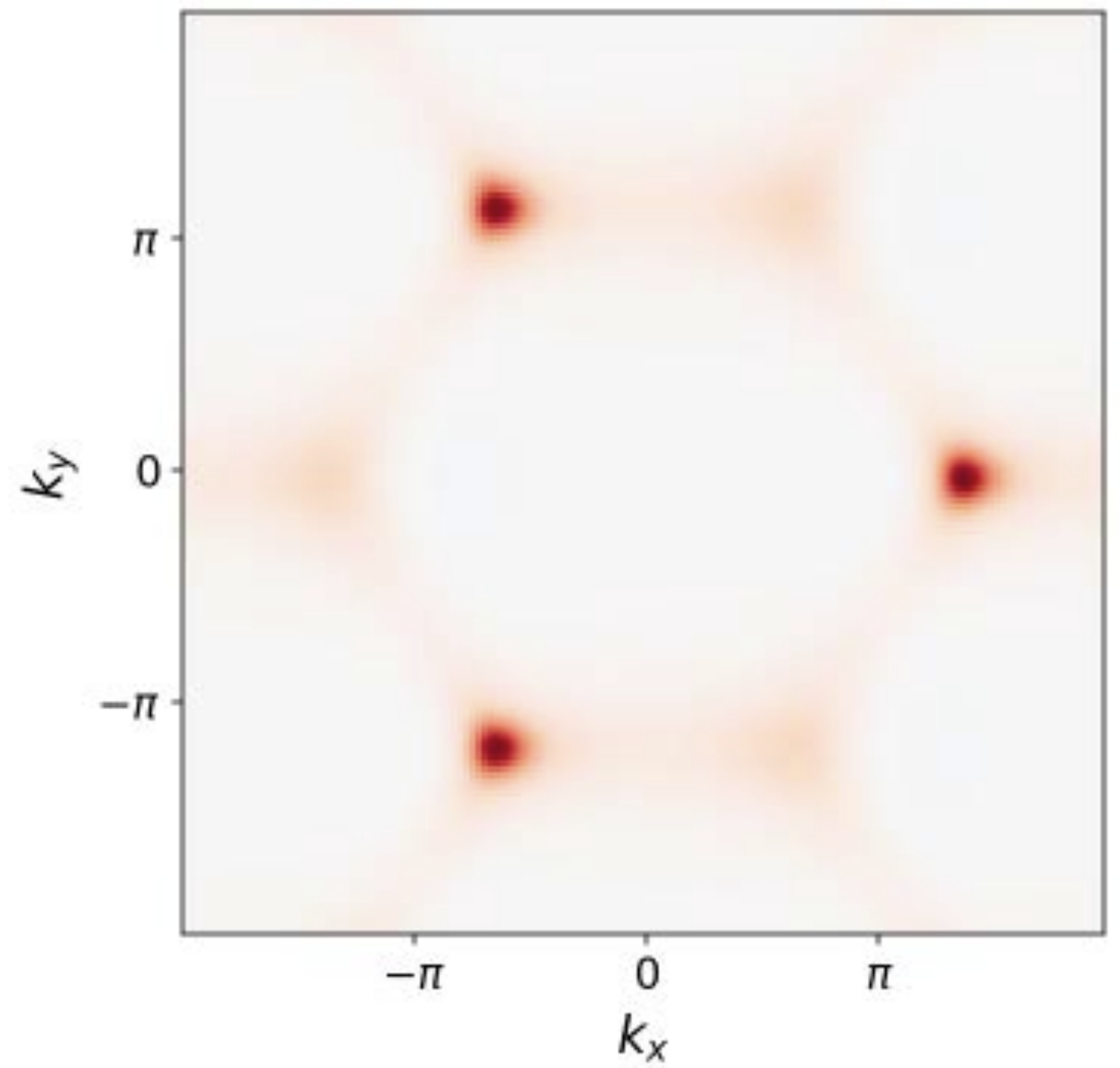


- Two of them are topological: non-trivial Chern numbers ± 1

- Chern number is a surface integral of Berry curvature:

  $\Rightarrow$ analogous of magnetic flux, which measures magnetic sources (monopoles) in Brillouin zone

$$C = \frac{1}{2\pi} \int_{\mathbf{k} \in BZ} \mathrm{d}^2\mathbf{k} \left(\nabla_{\mathbf{k}} \times \overrightarrow{A}_{n,\mathbf{k}} \right)$$



- Chern numbers can be probed through quantized Hall response and stable gapless edge modes

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General goal

- Theoretically **design topological matter** through accessible resources in a lab

  - As underlying degrees of freedom, we are interested in usual, non-topological, **superconducting wires**
  - Couple such wires through **Josephson junctions**
  - Break time-reversal symmetry with **magnetic flux**

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Superconductivity

- BCS s-wave superconductivity, where $c_{i\sigma}^{\dagger}$ is the electron creation operator and $n_{i\sigma} = c_{i\sigma}^{\dagger} c_{i\sigma}$

$$H = -t \sum_{\langle i,j \rangle, \sigma} c_{i\sigma}^{\dagger} c_{j\sigma} - U \sum_{i} n_{i\uparrow} n_{i\downarrow}$$

- In **mean field** approach, this reduces to the quadratic Hamiltonian in momentum space

$$\hat{H}_{\mathrm{MF}} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left(\Delta c_{\mathbf{k}\uparrow}^{\dagger} c_{-k\downarrow}^{\dagger} + \Delta^{*} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right)$$

Where $\Delta \equiv -U \sum_{\mathbf{k}} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle = |\Delta| e^{i\theta}$ is the **superconductor order parameter** so that

$$1 = U \sum_{\mathbf{k}'} \frac{1}{2E_{\mathbf{k}'}} \tanh\left(\frac{E_{\mathbf{k}'}}{2k_B T} \right) \qquad \text{(gap equation)}$$

with $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^{2} + |\Delta|^{2}}$

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Josephson Junctions

- The superconductor phase θ effectively captures the bosonic character of the condensate

- A Josephson junction (JJ) consists of a **barrier for Cooper pairs** to tunnel through (in fermion description this corresponds to a weak link Γ)

- In the presence of magnetic flux, the SC order parameter shift contributes to the system energy

$$H = -E_J \cos(\theta_1 - \theta_2 - \phi), \qquad E_J \sim \frac{\Gamma^2}{|\Delta|}$$

- **Josephson current** through the junction (with no voltage applied)

$$I = I_0 \sin(\theta_1 - \theta_2 - \phi)$$

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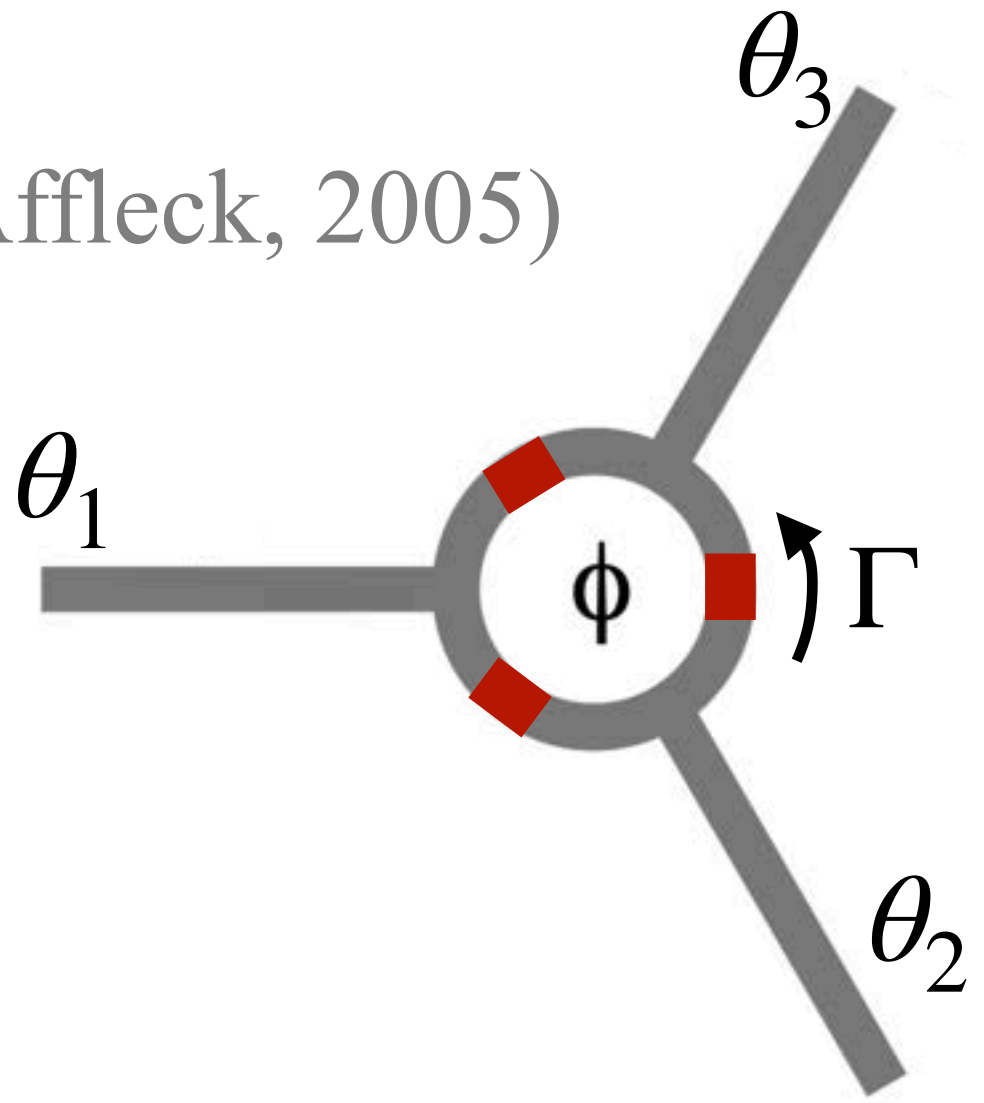
Splitters / Circulators

- We now have the basic ingredients to introduce the **Cooper-pair splitter**
- Connecting Y-junctions with JJs and adding flux
- Explored in the context of quantum wires (Oshikawa, Chamon & Affleck, 2005)

$\Rightarrow$ Three terminals circulator with flux ϕ

$\Rightarrow$ Josephson Junctions Γ



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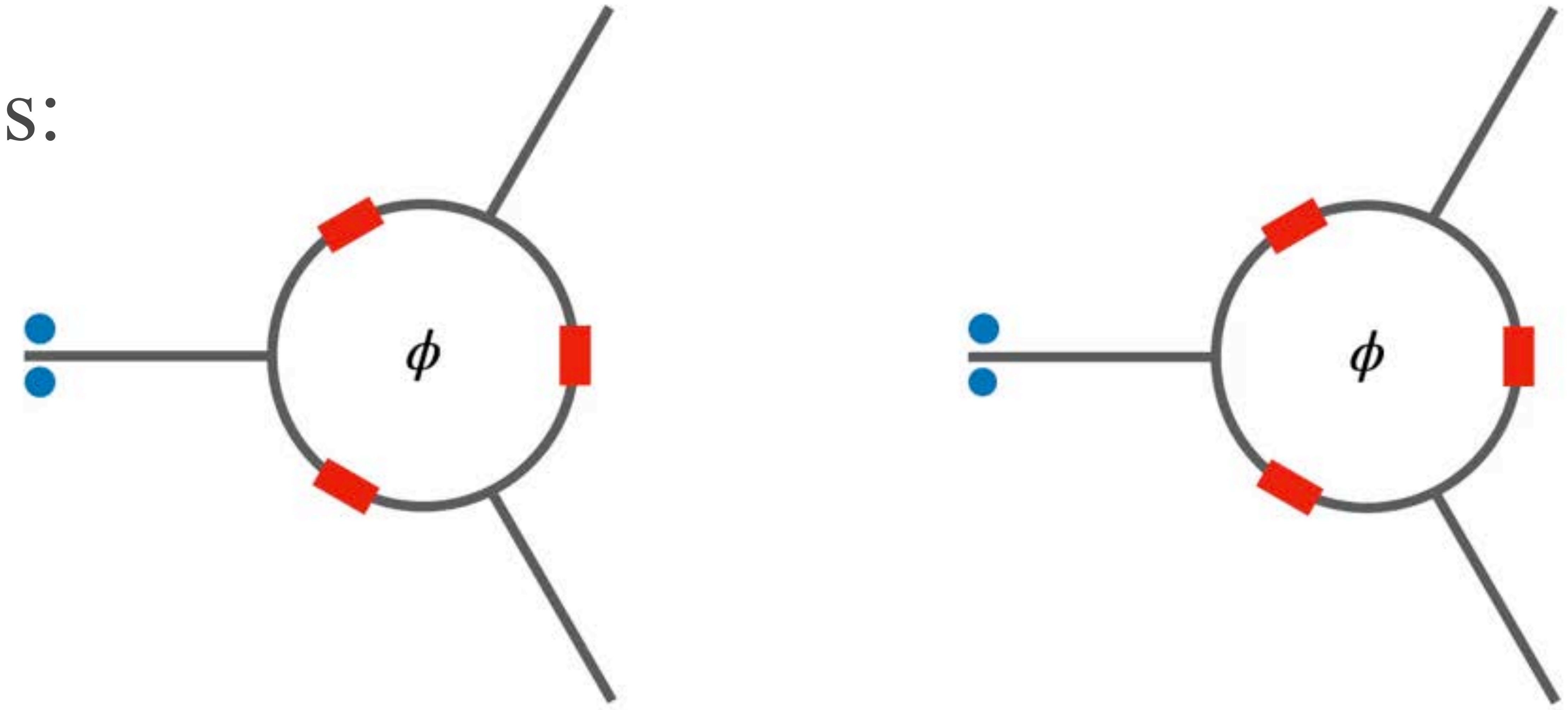
Transport processes

- There are two competing transport processes:

  ⇒ Charge carrier is a **Cooper pair**

  ⇒ Charge carrier is a **single electron**



- Consider magnetic flux $\phi = \frac{1}{2}(2n+1)\pi$, (in units of flux quanta $\Phi_0 = h/2e$)

- The Arahonov Bhom phase 2ϕ is **destructive for the pair**, but **non-destructive for single electrons**

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Large coherence length

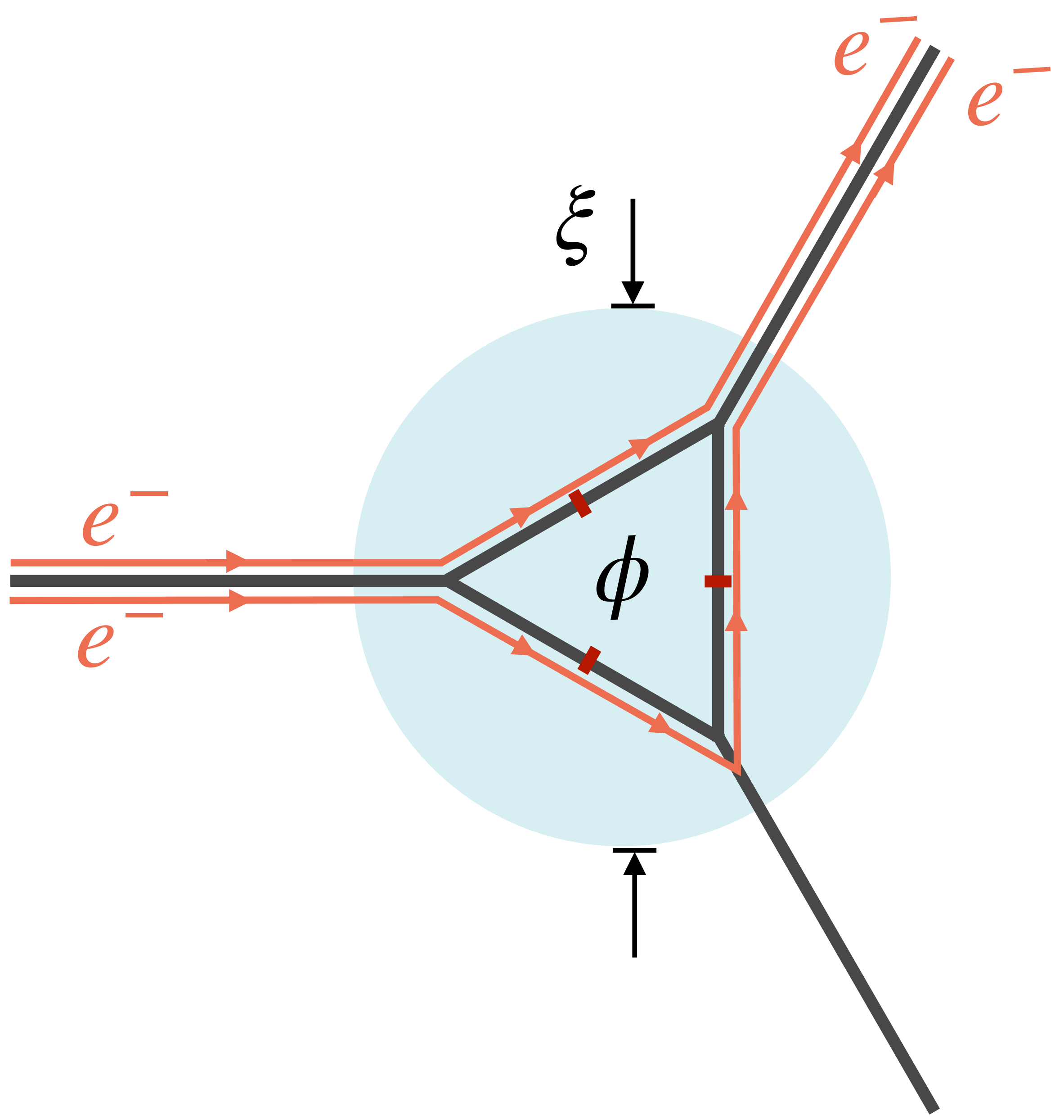
- The superconducting gap Δ induces a coherence length,

$$\xi \sim \frac{1}{\Delta}$$

This is a measure of how far away electrons in a Cooper-pair can separate from each other

⇒ Large superconducting gap implies Cooper-pairs traveling tightly together: no splitting

⇒ We expect to see Cooper-pair splitting for **large enough** ξ (order of size of circulator)



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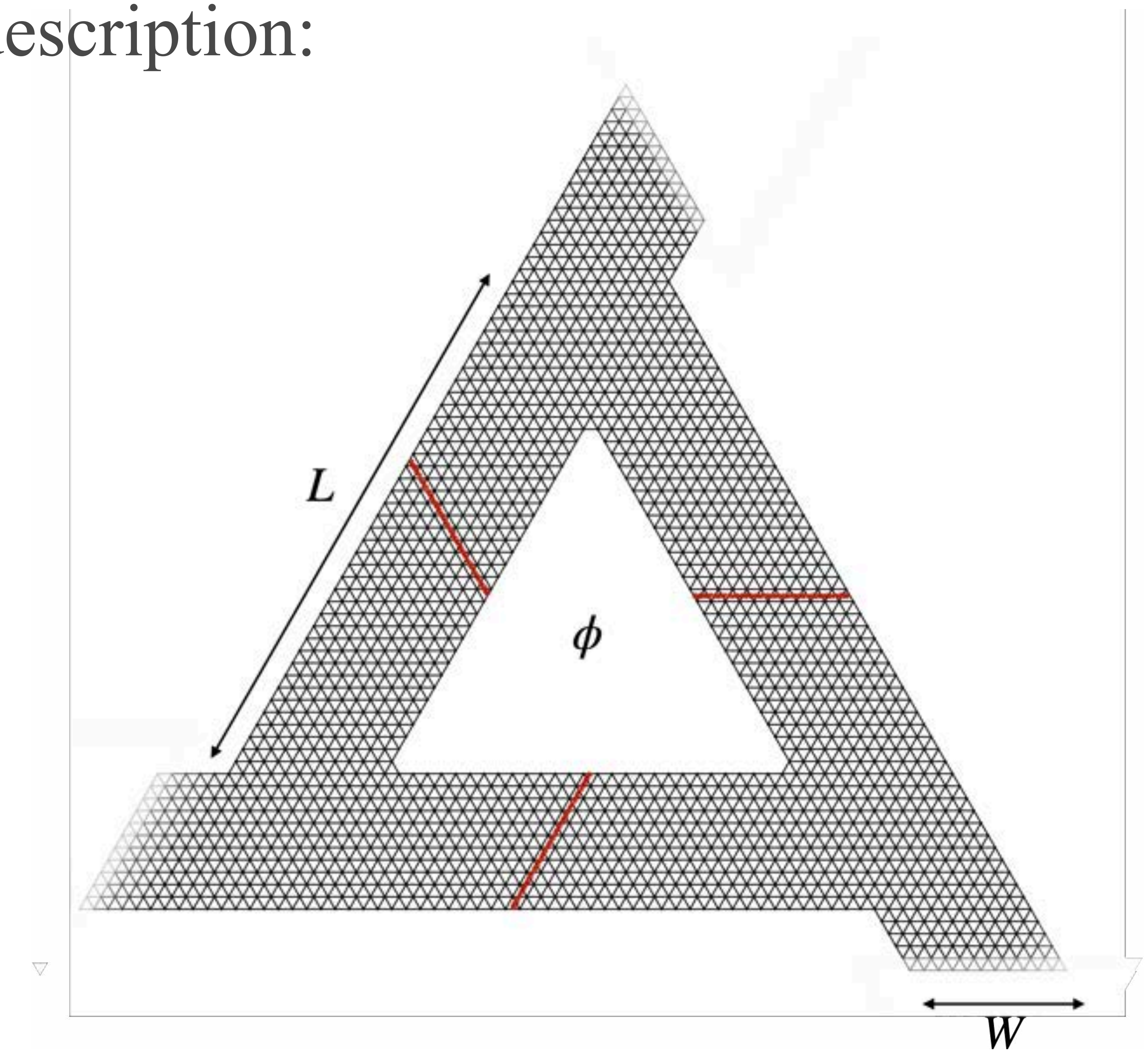
Numerics - Kwant

- Describe the circulator in terms of an effective, mean-field, description:

$$H = H_t + H_\Gamma + H_\Delta$$



where:

$\Rightarrow$ H_t contains **intra-wire** electron hopping

$\Rightarrow$ H_Γ contains **inter-wire** electron (Josephson weak link)

$\Rightarrow$ H_Δ contains **superconductivity** pairing terms

Numerical results through exact diagonalization and Kwant (Groth, Wimmer, Akhmerov, & Waintal, 2014)

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Physics of a circulator

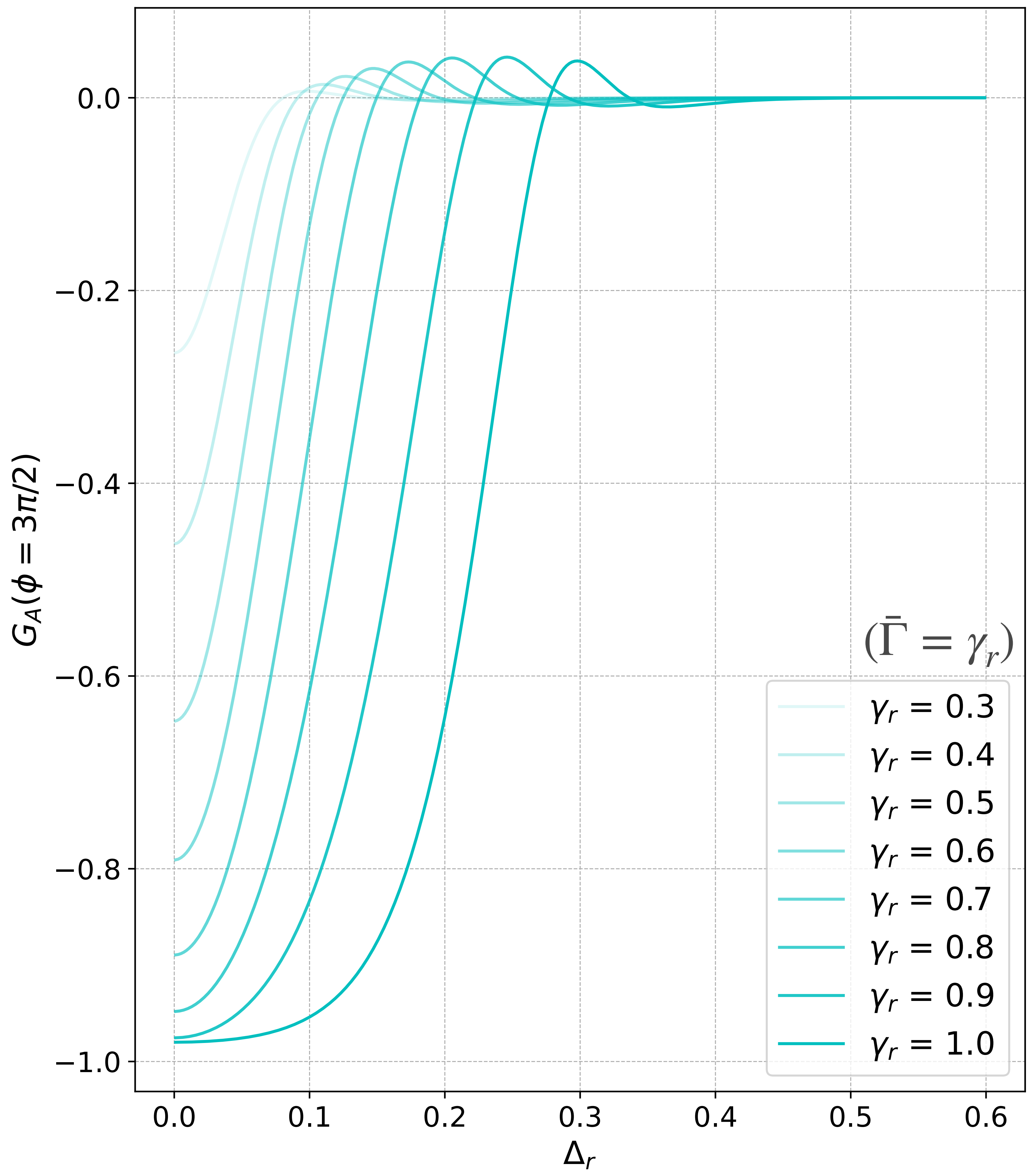
- Cooper-pair splitting can be captured by the **non-local antisymmetric conductance** (Wang & Hu, 2011)

$$G_A = \frac{1}{3}\epsilon_{ij} G_{ij}$$

- Let L the size of circulator and *a* lattice spacing. We define the dimensionless ratios $\bar{\Gamma} = \Gamma/t, \quad \bar{\Delta} = \Delta/t, \quad \bar{L} = L/a.$

- For semi-integer $\phi = \frac{1}{2}(2n+1)\pi$, we find that

$$\frac{G_A(\bar{\Gamma}, \bar{\Delta}, \bar{L})}{e^2/h} = \frac{16\,\bar{\Gamma}^3}{(1+3\bar{\Gamma}^2)^2}\, f\!\left(\frac{\bar{\Delta}}{\bar{L}^{\alpha}\bar{\Gamma}^{\beta}}\right)\exp(-\bar{\Delta}\bar{L})$$



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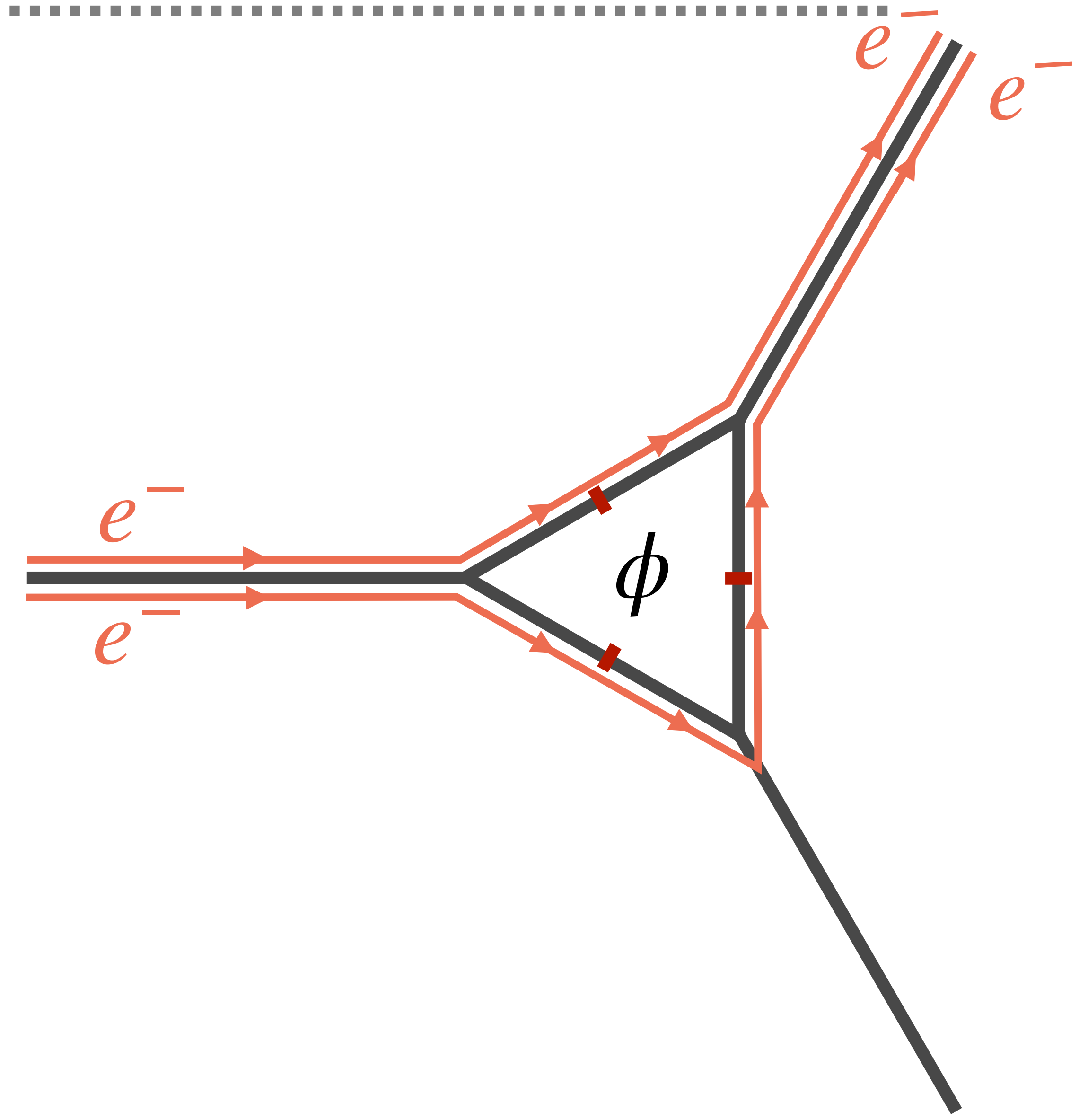
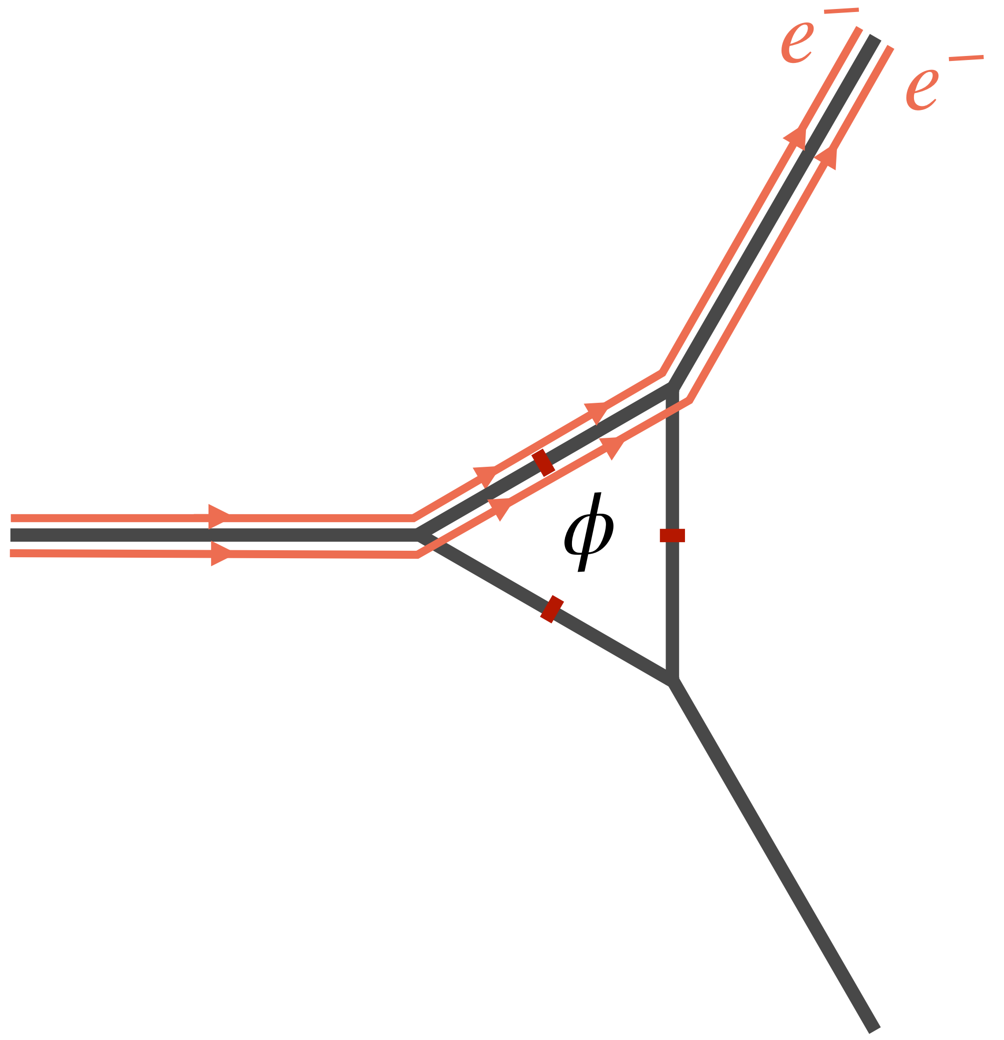
Physics of a circulator



- The function $f(x)$ increases to a peak at $x \approx 2$ and then decays to zero

$$\frac{G_A(\bar{\Gamma}, \bar{\Delta}, \bar{L})}{e^2/h} = \frac{16\,\bar{\Gamma}^3}{(1+3\bar{\Gamma}^2)^2} f\left(\frac{\bar{\Delta}}{\bar{L}^\alpha \bar{\Gamma}^\beta}\right) \exp(-\bar{\Delta}\bar{L})$$

- Cubic scaling with Γ suggests a third order process, instead of order two



- The exponential dependence introduces a coherence length $\xi \sim \Delta^{-1}$

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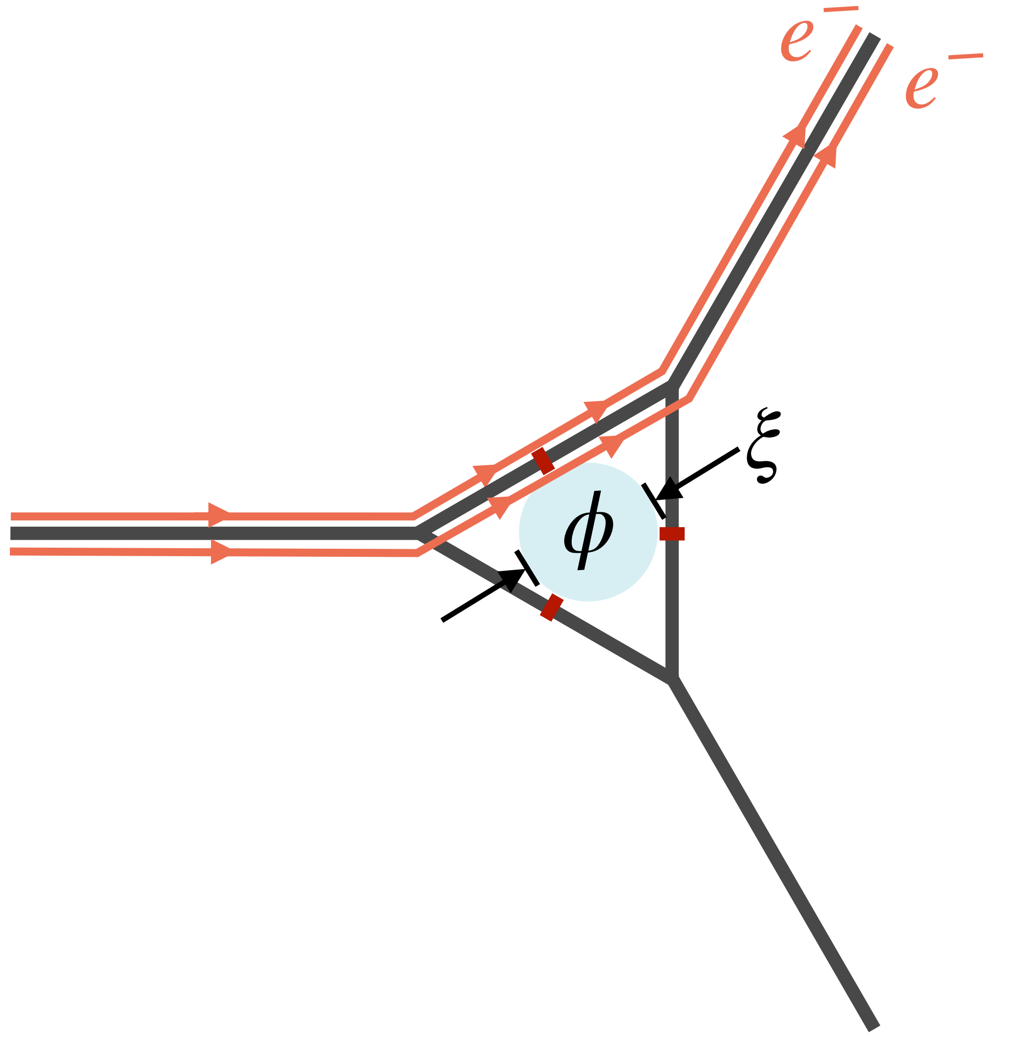
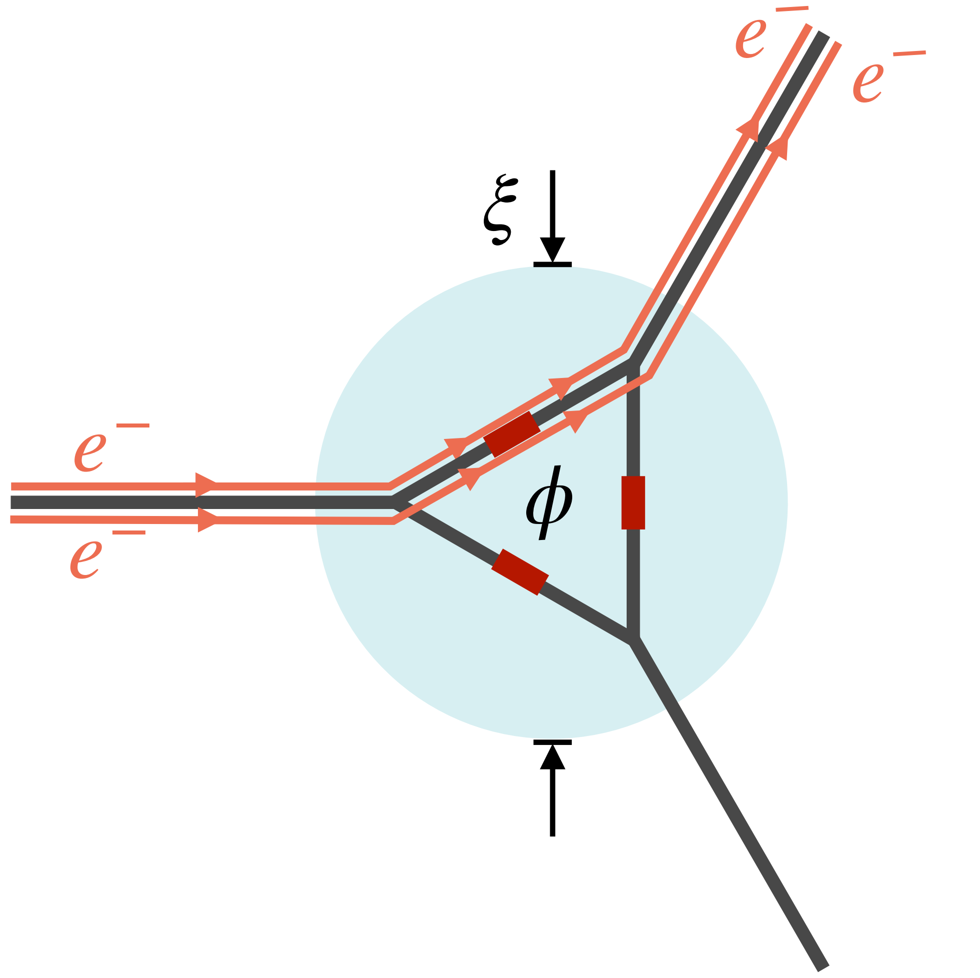
Splitting regimes

$$\frac{G_A(\bar{\Gamma}, \bar{\Delta}, \bar{L})}{e^2/h} = \frac{16\,\bar{\Gamma}^3}{(1+3\bar{\Gamma}^2)^2} f\left(\frac{\bar{\Delta}}{\bar{L}^{\alpha}\bar{\Gamma}^{\beta}}\right) \exp(-\bar{\Delta}\bar{L}),$$



- For **small coherence** length, $\bar{L} \gg \bar{\Delta}^{-1}$, the exponential decay dominates, causing $G_A \to 0$, reflecting the absence of pair splitting.

- For **barriers that are too thick**, $\bar{\Gamma}^{\beta} \ll \bar{\Delta}\bar{L}^{\alpha}$, the argument $x \gg 2$ of the function $f(x)$ becomes too large, placing the system outside the relevant pair-splitting regime.



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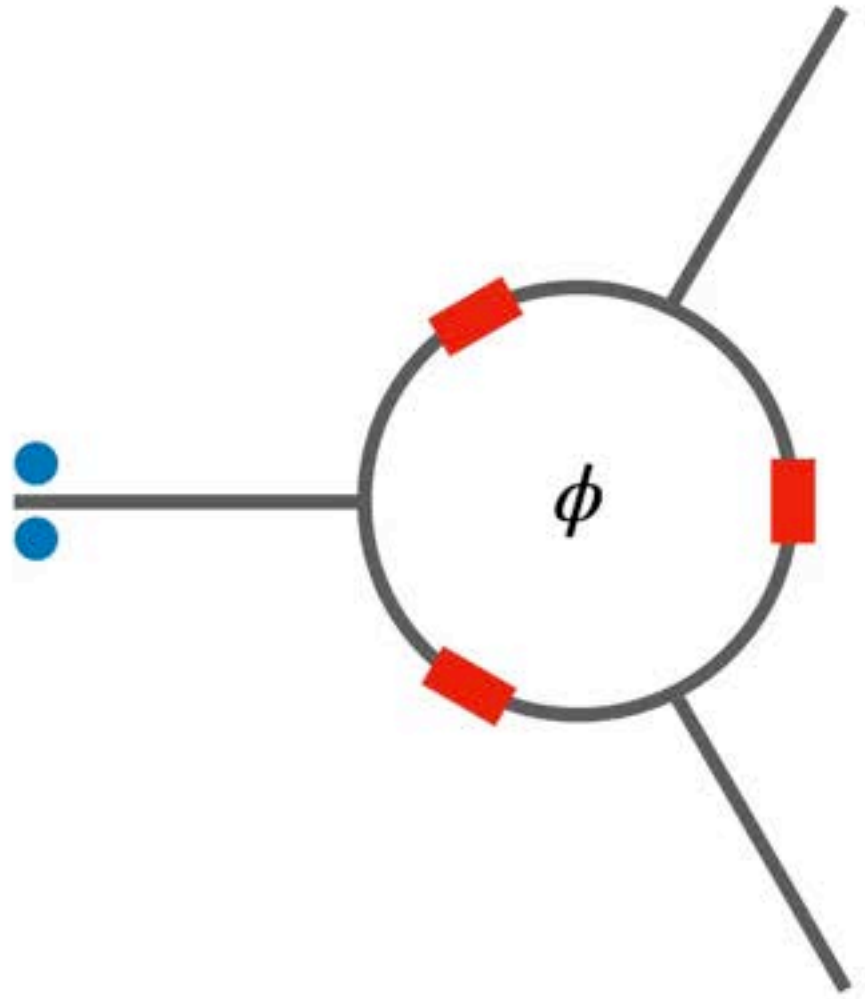
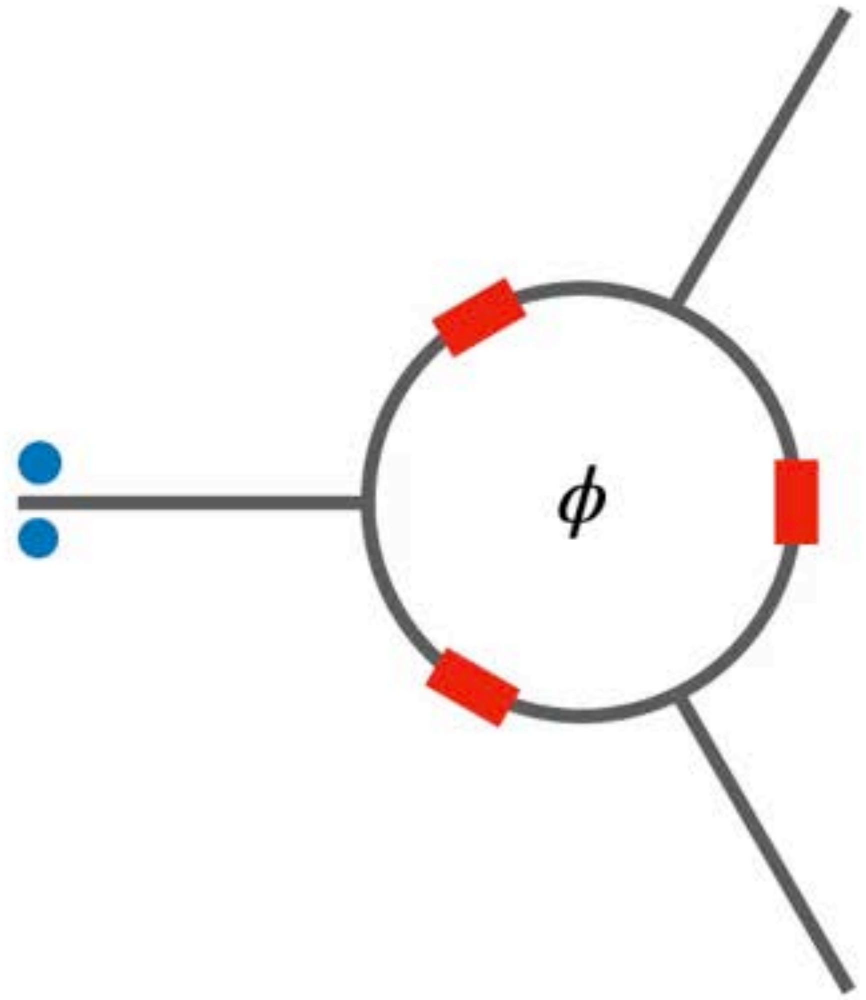
Takeaway

- In order to observe **splitting**, we need three key ingredients:
  - Semi integer multiple of π magnetic flux $\phi = \frac{1}{2}(2n+1)\pi$
  - Large coherence length $\xi \gg 1$
  - Barriers in Josephson junctions Γ cannot be too thick





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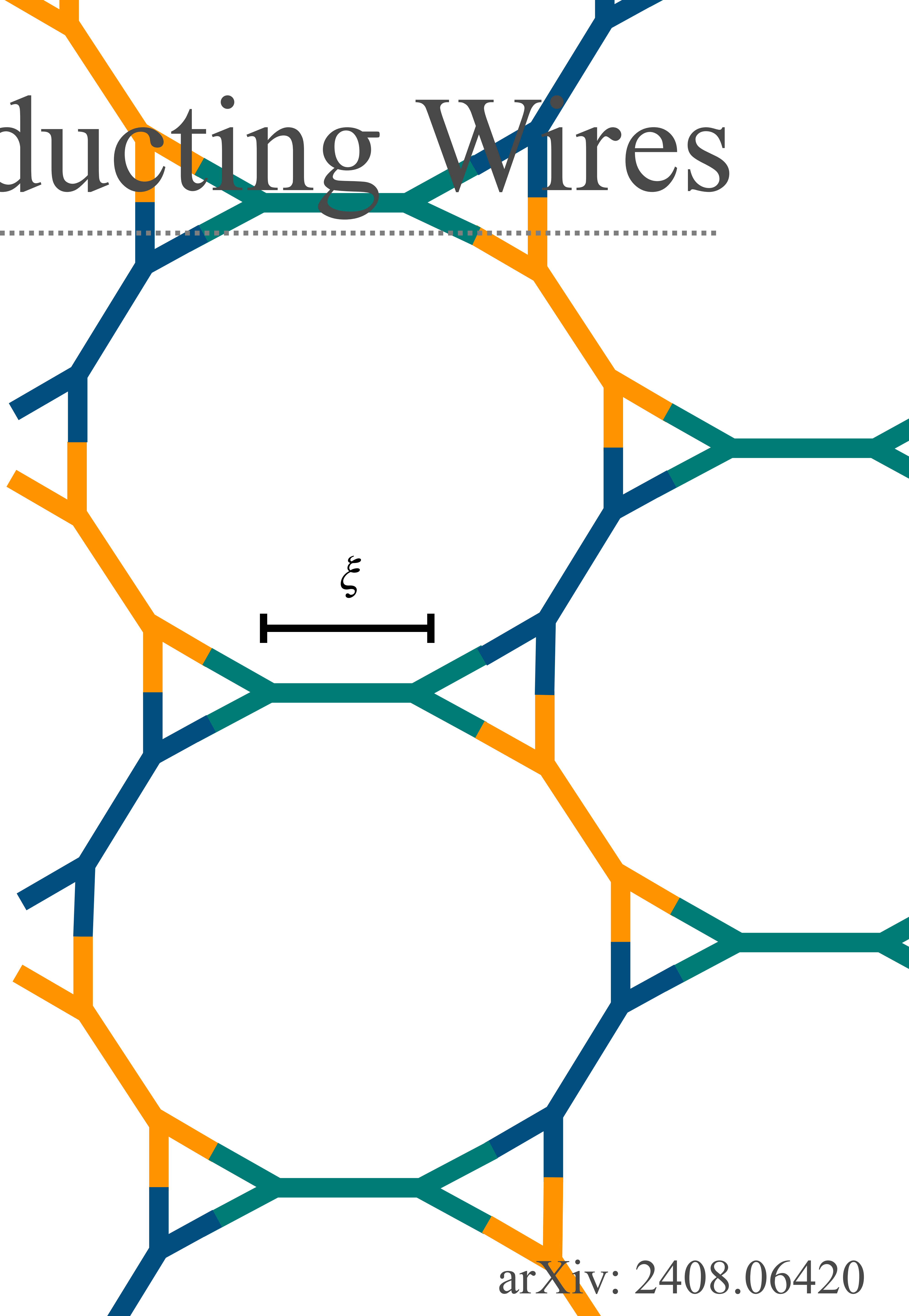
Networks of Superconducting Wires

- For large coherence length, each wire behave as a single degree of freedom
- **Meta-material:** array of mesoscopic wires
- Insert appropriate flux through the network
- Hope to see some sort of Chern insulator physics, but where the underlying dof are superconductors

$\Rightarrow$ **Topological superconductors** (?)



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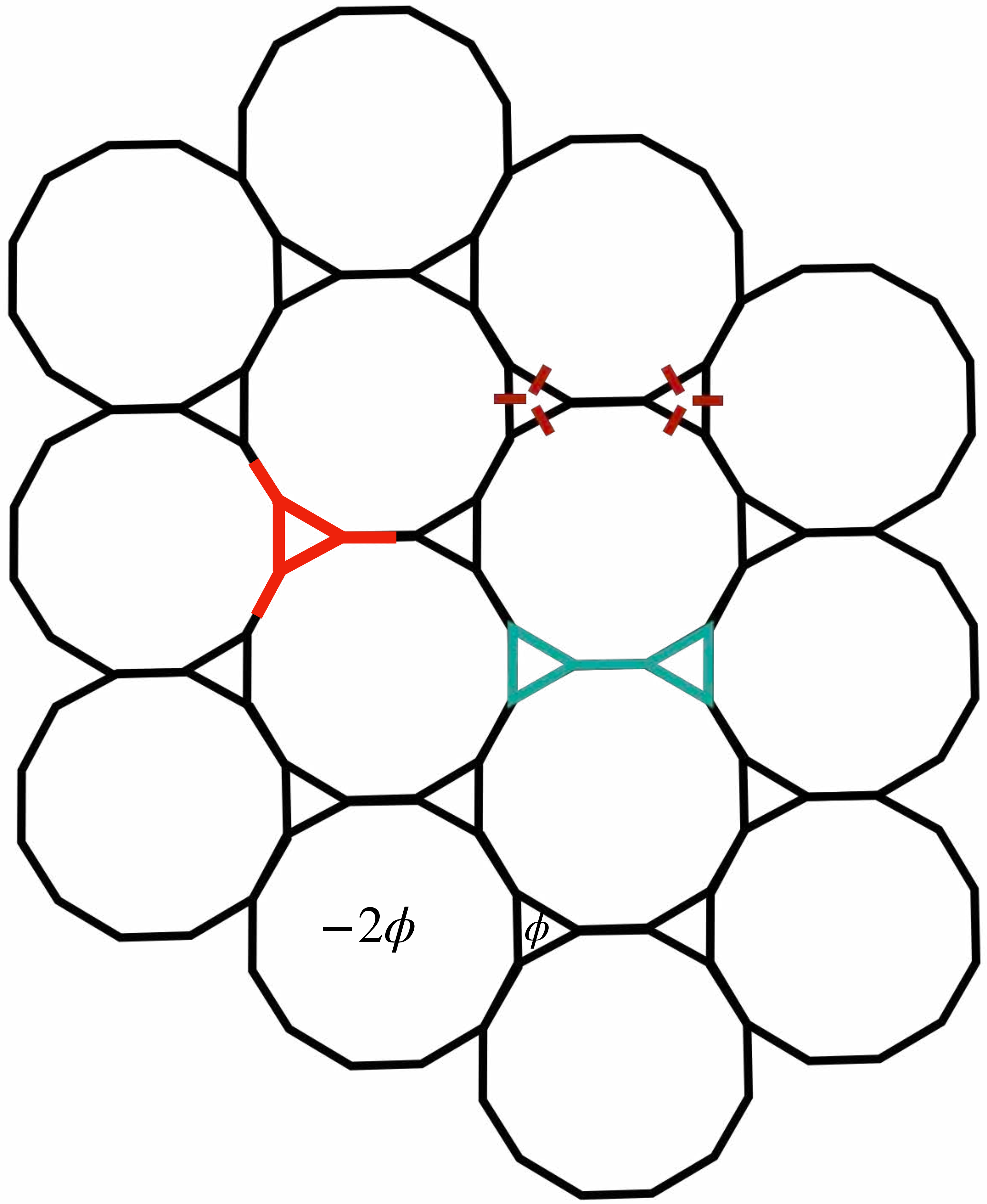
Network of circulators

- Network: Connect leads of Y-splitters to each other
- Archimedean $(3,12^2)$ geometry: triangles connected to dodecagons
- Flux of ϕ through triangles and -2ϕ through the dodecagons, which is only defined mod 2π
- Different arrangements can, in principle, lead to different behaviors of the system



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Effective Model

- Describe the network of wires in terms of an effective model:
- Large coherence length: approximate each wire as a single site

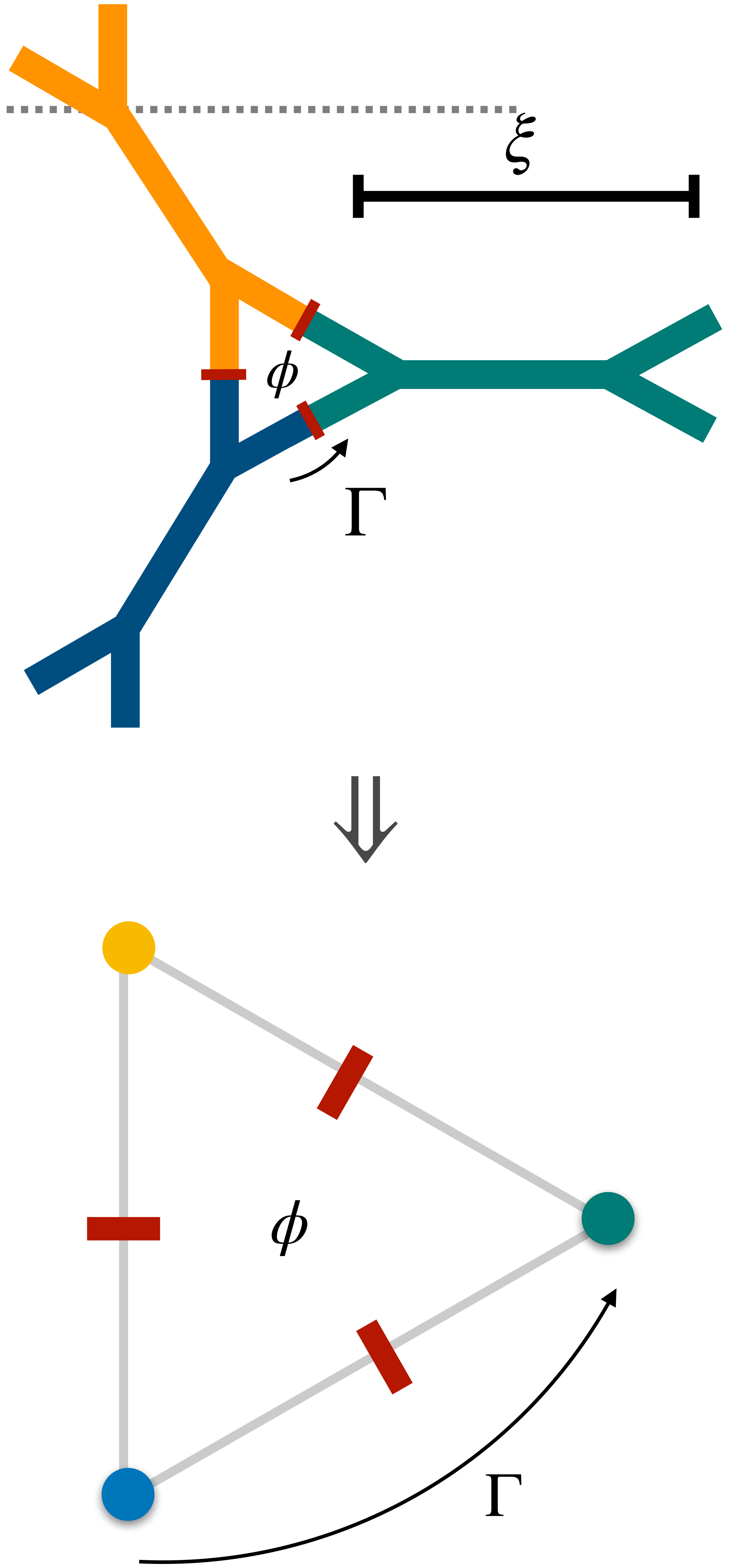
$$H = H_\Gamma + H_\Delta$$

where:

$\Rightarrow$ H_Γ contains electron hopping between neighboring sites

$\Rightarrow$ H_Δ contains superconductivity pairing terms

Intra-wire hopping terms H_t renormalize the effective parameters



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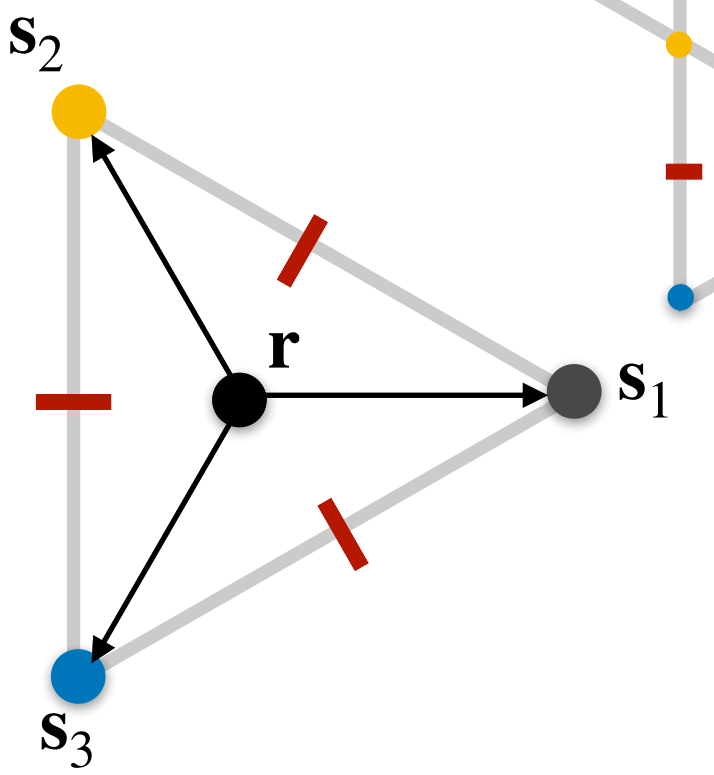
Ugly Hamiltonian

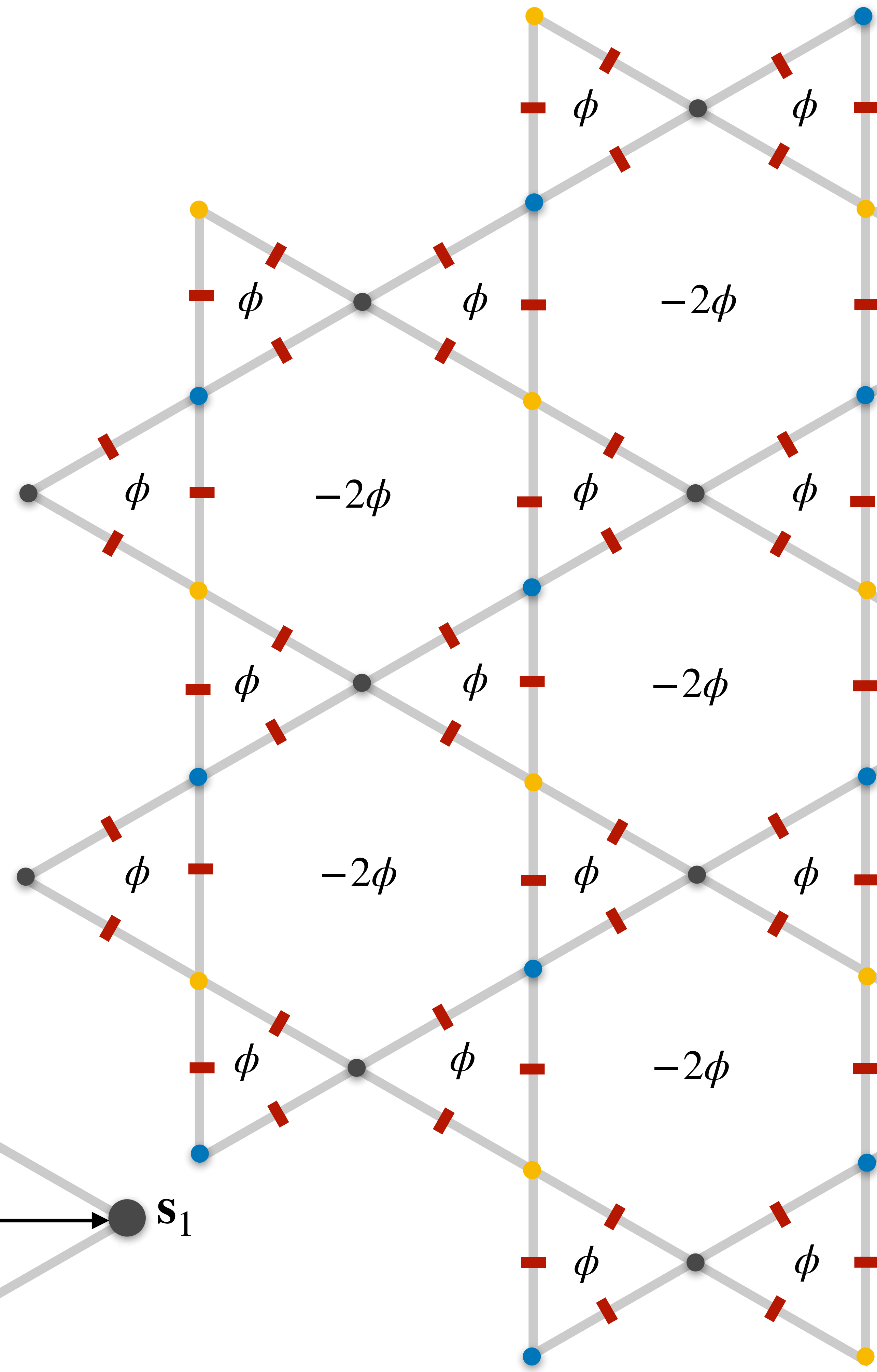
- The mean-field matrix Hamiltonian is $H_k = \begin{bmatrix} h_k & \Delta_k \\ \Delta_k^* & -h_{-k}^* \end{bmatrix}$, where

$$h_k = \begin{bmatrix} 0 & \alpha_\triangleright + \alpha_\triangleleft d_k^{12} & \bar{\alpha}_\triangleright + \bar{\alpha}_\triangleleft d_k^{13} \\ \bar{\alpha}_\triangleright + \bar{\alpha}_\triangleleft d_k^{21} & 0 & \alpha_\triangleright + \alpha_\triangleleft d_k^{23} \\ \alpha_\triangleright + \alpha_\triangleleft d_k^{31} & \bar{\alpha}_\triangleright + \bar{\alpha}_\triangleleft d_k^{32} & 0 \end{bmatrix}$$ is the hopping matrix,

$$\Delta_k = \Delta \begin{bmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{bmatrix}$$ is the superconducting matrix

In the above, $d_k^{ij} = e^{-ik\cdot(s_i - s_j)}$ and $\alpha_\triangleright = \alpha_\triangleleft \equiv \Gamma e^{i\phi/3}$





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Phase Diagram

- Express the phase diagram in terms of Γ/Δ and ϕ. For this, we :

  $\Rightarrow$ Compute the many-body **ground state energy** as a function of θ_a parameters

  $\Rightarrow$ Solve for the configurations θ_a^* **that minimize** the total energy

  $\Rightarrow$ Compute **Chern numbers** in momentum space

- Define chirality, $\chi \sim \sin(\Delta\theta)$ which indicates JJ current direction

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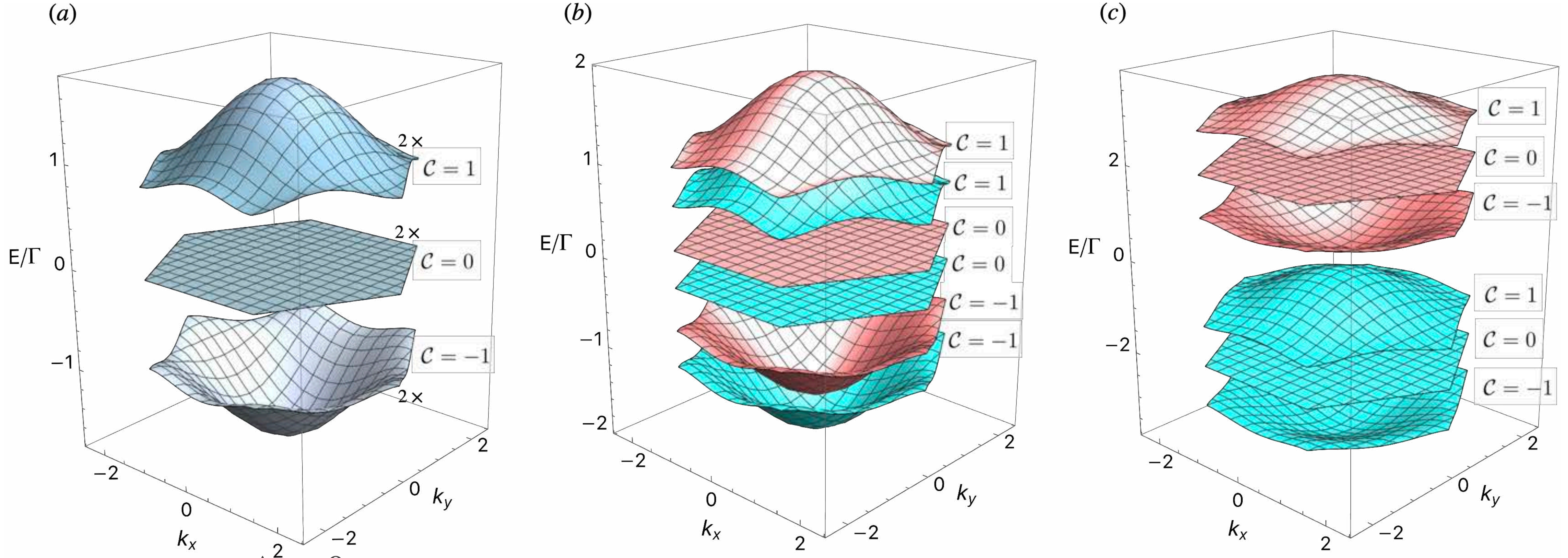
Exactly solvable case: $\phi = 3\pi/2$

- For $\phi = 3\pi/2$, numerically find that all $\theta_a = 0$. Then, energy bands are given by. $\pm\Delta$, $-\epsilon_{\mathbf{k}} \pm \Delta$, $\epsilon_{\mathbf{k}} \pm \Delta$, with

$$\varepsilon_{\mathbf{k}} = \frac{\Gamma}{\sqrt{2}}\sqrt{3 + \cos(\sqrt{3}k_x) + \cos\left(\frac{\sqrt{3}}{2}k_x - \frac{3}{2}k_y\right) + \cos\left(\frac{\sqrt{3}}{2}k_x + \frac{3}{2}k_y\right)}.$$



$\Delta = 0$ $\Gamma \gg \Delta$ $\Gamma \ll \Delta$

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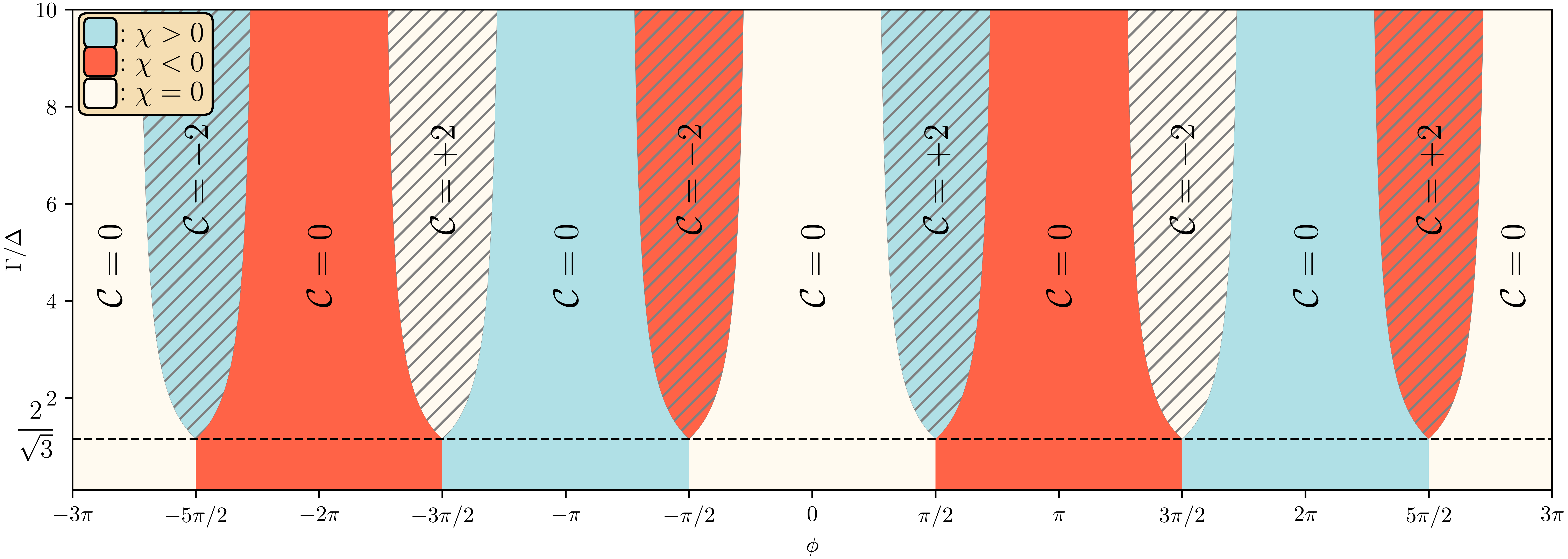
Phase Diagram



- Alternating stripes of trivial ($C = 0$) and **topological superconducting phases** ($C \neq 0$)
- **Critical value** $(\Gamma/\Delta)^* = 2/\sqrt{3}$: below such a value, all phases are topologically trivial
- Below the critical line $(\Gamma/\Delta)^*$, chirality exactly match classical configurations that minimize

$$H_{JJ} = -E_J\left[\cos\left(\theta_2 - \theta_1 + \frac{2\phi}{3}\right) + \cos\left(\theta_1 - \theta_3 + \frac{2\phi}{3}\right) + \cos\left(\theta_3 - \theta_2 + \frac{2\phi}{3}\right)\right].$$

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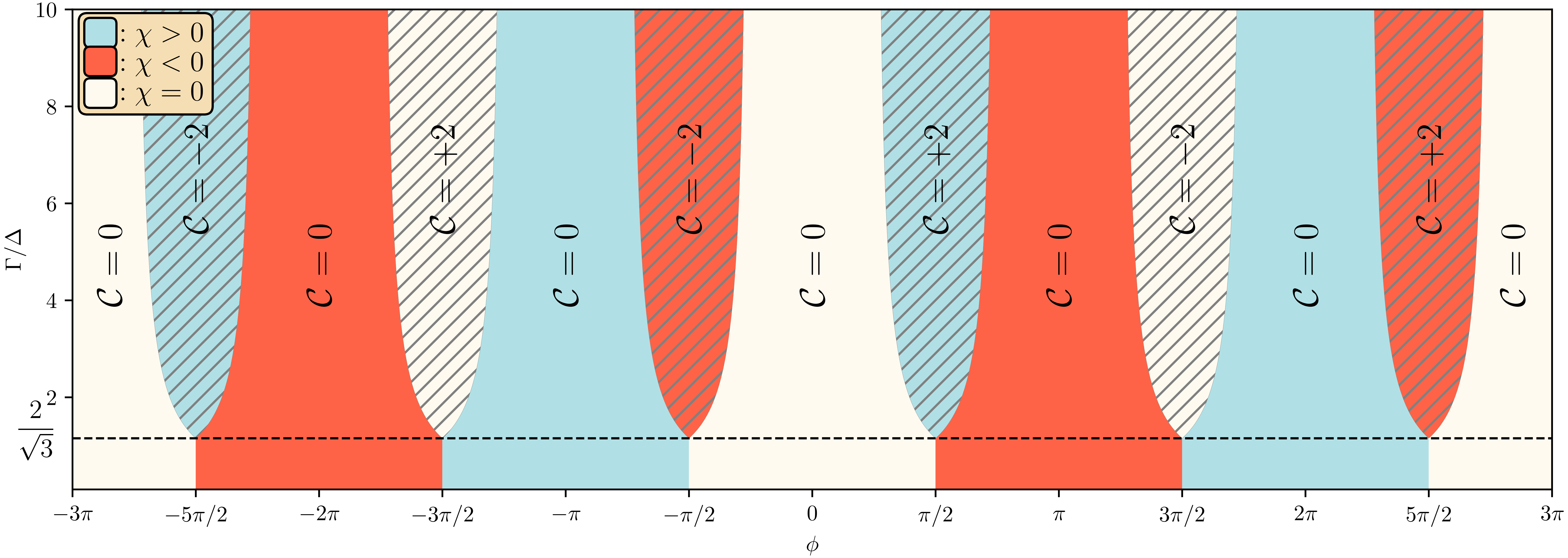
Phase Diagram



- For flux integer multiples of π time-reversal symmetry is restored: vanishing Chern-numbers
- Topological superconductors phases appear at **maximally breaking of time-reversal**: half-integer flux values ⇒ transport properties are dominated by **Cooper-pair splitting**
- The consecutive splitting of Cooper-pairs at each node builds up to robust topological properties

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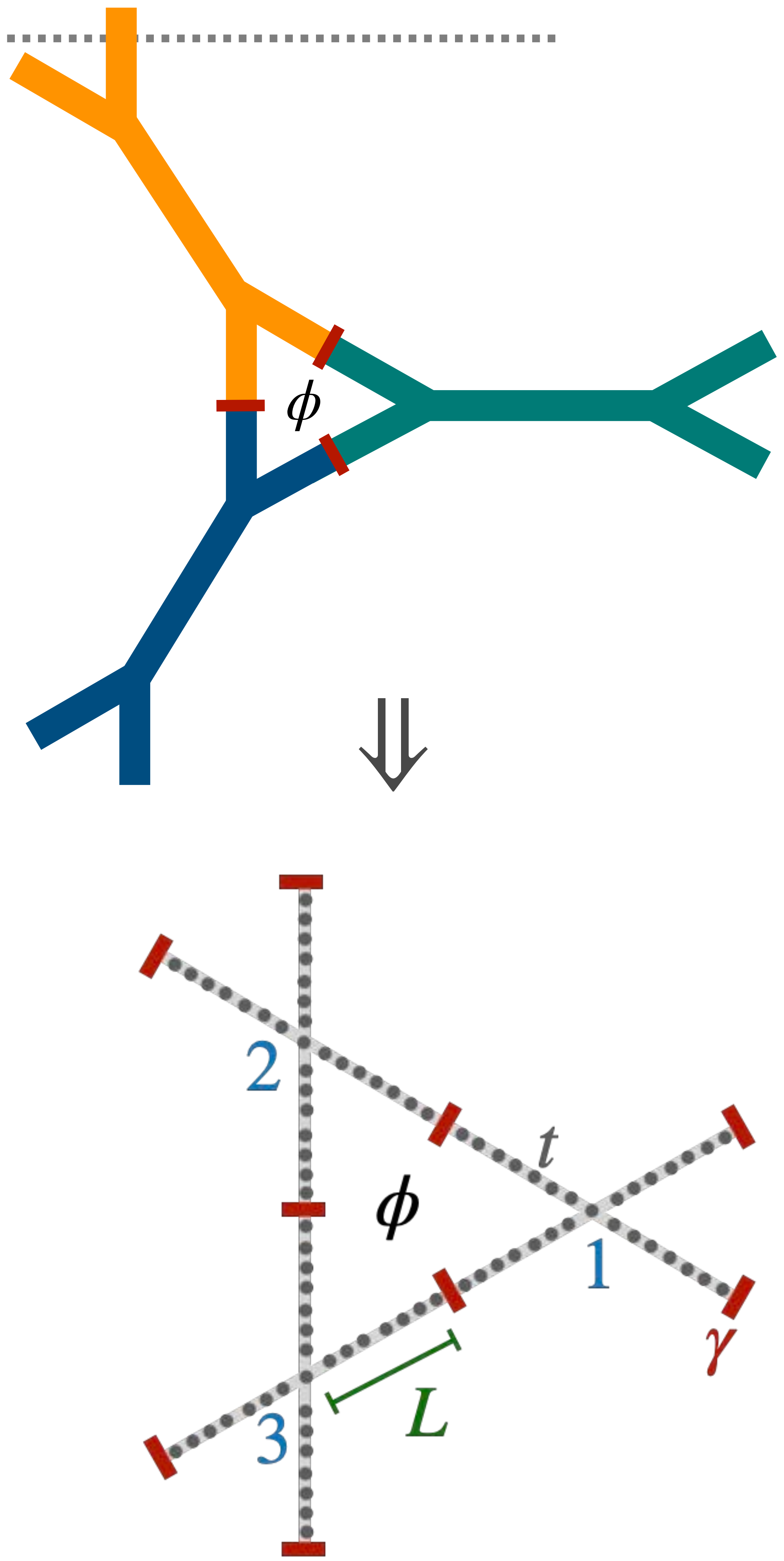
Microscopic model

- Later in the paper we **justify** our previous approximation

  ⇒ We consider each extensive wire as a set of 4L dots

- We analytically find all the orbital wave functions and numerically compute the energy bands and Chern numbers

⇒ The **results agree with our effective description**, a more intuitive picture



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Final Remarks

⇒ Cooper-pair splitting require a balance between **superconductivity, flux and JJ junctions**

⇒ Observe a direct relation between **pair splitting** and **topological superconductivity** in a network of circulators

Future Perspectives:

- Explore more general networks of hybrid circuit elements
- Design richer (non-Abelian, fracton, etc) topological phases (?)

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Thank you!

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Parameters

- Aluminum $\xi \sim 1.6\mu m$
- Typical sizes of wires: $\sim$ 500 nm
  - Arrays with 1000 x 1000 (Nb) wires (Sohn et al, 1993)
- Temperature $T \sim 1.2K$
- Critical magnetic field $H_c \sim 0.01T$
  (enough to get us desired ϕ values)

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